Journal of Computational Science & Engineering 63 (2023) 1-6

Available at http://www.asocse.org ISSN 1710-4068

# Dynamic simulation of spatially deployable structures based on Lie group modeling

Wensong Liang[1], Yuan Yue[1], Yang Xue[2], Jieyu Ding[1*]

[1]School of Mathematics and Statistics, Qingdao University, Qingdao 266071, China

[2]School of Business, Qingdao University, Qingdao 266071, China

**Abstract:** Aiming at spatially deployable structures in multi-body systems, the modeling of spatially deployable structures based on Lie group method is studied. Runge-Kutta method is used to model and calculate the typical deployable structures, and the dynamic simulation results with high accuracy are obtained. The Lie group method maintains the orthogonality of the rotation matrix while maintaining the constraint equation, that is, the system structure.

**Keywords:** Dynamics of multibody system; Developable structures; Differential-algebraic equations; Lie groups

Received 15 November 2023, Received 21 November 2023, Received 28 November 2023

*Corresponding Author: Jieyu Ding, djy@qdu.edu.cn

## 1. Introduction

Deployable structure is a new type of aerospace structure that was born with the development of aerospace science and technology in the late 1960s. China's deployable space structure has been successfully applied or is being applied to a number of space projects such as satellite system solar cells, large aperture spaceborne antennas, space station extension arms, satellite communication system solar wings, rigid and flexible solar wings, Mars and moon rovers. Large-scale, lightweight and high-precision space deployable structures have become a hot topic in the field of aerospace science and technology in China.

Common deployable structures are mainly divided into shear hinge structure, space deployable film plane structure, shear deployable structure, double deployable scissors structure, space hinged deployable truss structure and so on. The main modeling method for developable structures is finite element method. As a special multibody system, the modeling methods of developable structure include natural coordinate method, Euler Angle method, Euler quaternion method, Lie group Lie algebra method, etc. The Lie group Lie algebraic method uses exponential mapping to iterate, so that the rigid body attitude rotation matrix is always maintained as orthogonal matrix. A special orthogonal group is used to describe the attitude of the object so as to avoid the strange phenomenon caused by the parametric expression, and the inherent Lie group property of the attitude of the object is always maintained in the calculation process. It is beneficial to improve the computing efficiency.

In this paper, the Lie group method is used to analyze the deployable structures, the differential algebraic equations are established, and some special kinds of deployable structures are numerically calculated to study whether their rotation matrices maintain the orthogonality of Lie groups and the constrained stability.

## 2. Lie Group theory

Definition 1.1 The set G is a Lie group if it represents both a group and a differential manifold, and the mapping $F: G \times G \to G$, $F(x,y) = xy^{-1}$ is $C^{\infty}$.

Definition 2.2 for a continuously differentiable function $q(t), q(t_0) \in G$ if the derived function $\dot{q}(t) \in T_qG$, $q = q(t): \dot{q}(t) \in T_{q(t)}G$, $(t > t_0)$, has the $q(t) \in G$ .In particular, for the unit e, $T_eG$ is the Lie algebra g ,when $q = e$.

$\forall q \in G$, every element of $T_qG$ can be expressed by the element $\tilde{v}$ in the Lie algebra. By defining the bijection $L_q: G \to G, y \mapsto L_q(y) := q \circ y$, its derivative $\mathrm{D}L_q$ at $y = e$ can correspond to the mapping relationship between the Lie algebra and the tangent space:

$$T_qG = \{\mathrm{D}L_q(e) \cdot \tilde{v}: \tilde{v} \in g\} = \{\mathrm{D}L_q(e) \cdot \tilde{v}: v \in R^k\} \tag{1}$$

The following form can be obtained:

$$\dot{q}(t) = \mathrm{D}L_{q(t)}(e) \cdot \tilde{v}(t) \tag{2}$$

Where $v(t) \in R^k$, in formula (9), $L_q$ and tilde $(\tilde{\bullet})$ depend on the Lie group. For velocity $v$, equation (2) is locally satisfied

$$q(t) = q(t_0) \circ exp((t - t_0)\tilde{v}) \in G \tag{3}$$

And the exponential mapping: $g \to G$, for Lie group matrices, the exponential mapping is:

$$exp(\tilde{v}) = \sum_{i=0}^{\infty} \frac{1}{i!} \tilde{v}^i \tag{4}$$

Formula (4) partial differential homeomorphism, namely for $\forall q_a \in G$, $\exists$ neighborhood $U_{q_a} \subset G$, $\exists V_{\tilde{o}} \subset g$, making$\forall q \in U_{q_a}$, there is the only $\tilde{\Delta} q \in V_{\tilde{o}}$, a $q = q_a \circ exp(\tilde{\Delta} q)$was set up, An approximate form of equation (4) can be obtained by Rodrigues' formula

$$exp_{SO(3)}(\tilde{\Omega}) = I_3 + \frac{sin\|\Omega\|_2}{\|\Omega\|_2}\tilde{\Omega} + \frac{1-cos\|\Omega\|_2}{[\|\Omega\|_2]^2}\tilde{\Omega}^2 \tag{5}$$

Formula (3) can also be expressed as $v = T(q)\dot{q}(t)$, $T(q)$ is the tangent operator of exponential mapping (4), corresponding to equation (5), the formula is given

$$T_{SO(3)}(\tilde{\Omega}) = I_3 + \frac{cos\|\Omega\|_2 - 1}{[\|\Omega\|_2]^2}\tilde{\Omega} + \frac{1-\frac{sin\|\Omega\|_2}{\|\Omega\|_2}}{[\|\Omega\|_2]^2}\tilde{\Omega}^2 \tag{6}$$

In inertial coordinates, the position coordinates of a single rigid body in a rigid body system are usually expressed only by the vector $x \in R^3$ in linear space. The expression of rigid body attitude and its finite rotation is prone to singularity, but in Lie group method, the expression of Euler parameters including quaternion and rotation matrix can effectively avoid singularity. The expression of the rotation matrix $R$ is:

$$R \in SO(3) := \{R \in R^{3\times3} : R^T R = I_3, det\, R = +1\}. \tag{7}$$

The set $SO(3)$ as a three-dimensional differential manifold on $R^{3\times3}$, is combined with the linear space $R^3$ to represent this rigid body system using the element $q := (R, x)$ in the 6-dimensional group $G$, which can be combined in two ways:

In direct product $G = SO(3) \times R^3$, the group of operator $\circ$ defined as

$$(R_a, \mathrm{x}_a) \circ (R_b, x_b) = (R_a R_b, x_a + x_b) \tag{8}$$

The structural relation is obtained

$$\dot{R} = R\tilde{\Omega}, \dot{x} = u \tag{9}$$

where $u \in R^3$ represents the translational velocity in the inertial coordinate system, $\Omega = (\Omega_1, \Omega_2, \Omega_3)^T \in R^3$ represents the angular velocity, and has an antisymmetric matrix

$$\tilde{\Omega} = \begin{bmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{bmatrix} \in R^{3\times3} \tag{10}$$

The semi-direct product $G = SO(3) \times R^3$ can be seen as the special Euclidean group $SE(3)$, where the group operator$\circ$is defined as

$$(R_a, x_a) \circ (R_b, x_b) = (R_a R_b, x_a + R_a x_b) \tag{11}$$

The structural relation is obtained

$$\dot{R} = R\tilde{\Omega}, \dot{x} = RU \tag{12}$$

where $U \in R^3$ represents the translation velocity in a connected coordinate system.

For group elements $q = (R, x)$, the group operators in $G = SO(3) \times R^3$ and $SE(3)$ are equivalent to matrix products of non-singular block structures in $\mathrm{R}^{7\times7}$ and $R^{4\times4}$, respectively, i.e

$$SO(3) \times R^3 : \begin{bmatrix} R & 0_{3\times3} & 0_{3\times1} \\ 0_{3\times3} & I_3 & x \\ 0_{1\times3} & 0_{1\times3} & 1 \end{bmatrix}, SE(3) : \begin{bmatrix} R & x \\ 0_{1\times3} & 1 \end{bmatrix} \tag{13}$$

Thus, when $r$ takes $A$ suitable value, $\mathrm{G} = SO(3) \times R^3$, $SE(3)$, and the group containing all rotation matrices $R$ are isomorphic, respectively, to $A$ subgroup of the generalized linear group $GL(r) = \{A \in \mathrm{R}^{r\times r} : det\,\mathrm{A} \neq 0\}$. From the block structure of the matrix and the orthogonal condition $R^T R = I_3$, $\mathrm{G} = SO(3) \times R^3$, $SE(3)$, $SO(3)$are isomorphic to differential manifolds in $GL(7), GL(4), GL(3)$ respectively.

## 3. Modeling of spatially deployable structural Lie groups

According to the spatial developable structure model, the characteristics of the model are analyzed, and the Lie group modeling of the local developable structure unit is carried out.

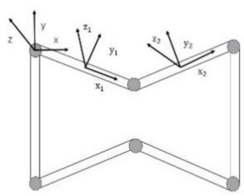

**Figure 1. Local model of developable structure**

Set up the coordinate system according to figure (1), take the midpoint of the bar in the figure as the node, and take the generalized coordinates as $\boldsymbol{q}_i = (x_i\ R_i), \boldsymbol{x_i} = [x_i\ , y_i, z_i]^T, i = 1,2 \ldots 6,$generalized mass matrix $\boldsymbol{M}_i = diag(m_i,\ J_i)$, $\boldsymbol{m}_i = diag(m_1, m_2, m_3), \boldsymbol{J}_i = (J_{i1}, J_{i2}, J_{i3}), i = 1,2 \ldots 6,$.

An expandable dynamic model is established based on analytical mechanics, and the Lagrange equation method is used to model the space expandable structure. The generalized coordinates are selected to establish the dynamic differential equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = F(q, V, t) \tag{14}$$

Where, the pull quantity $L$ is the difference between the system kinetic energy $T$ and the system potential energy $V$, that is, $L = T - V$; $F$is the generalized force of the system, which is usually a function of time and generalized coordinates, generalized velocity, and generalized acceleration.

$$T_i = \frac{1}{2}\dot{q}_i{}^T \boldsymbol{M}_i \dot{q}_i,\ \ i = 1,2 \ldots 6, \tag{15}$$

The total kinetic energy of the whole deployable unit is

$$T = \sum_{i=1}^{6} \boldsymbol{T}_i \tag{16}$$

The Lagrange function is

$$L = T - V \tag{17}$$

The non-potential generalized force on the whole deployable element is $F$.

By substituting equations (16) and (17) into equation (14), the differential equation is obtained:

$$\boldsymbol{M}(q)\ddot{\boldsymbol{q}} = \boldsymbol{F} \tag{18}$$

For a $K$-dimensional configuration space G containing a Lie group structure, the structural relation is given by equation (2), and the position coordinate$\mathbf{q}(t) \in$ G and the velocity vector $\mathbf{v}(t) \in R^k$.

Consider with $m \le k$ linear independent integrity constraints $\boldsymbol{\Phi}(\mathrm{q}) = [\Phi_1(\mathrm{q}), \Phi_2(\mathrm{q}) \ldots \Phi_m(\mathrm{q})]^T = 0$ dynamics system. By binding $\boldsymbol{B^T}(q)\boldsymbol{\lambda}$ coupled to the equilibrium equation of force and momentum. Here $\lambda(t) \in \mathrm{R}^m$ said though Lagrange multiplier, $\mathbf{B}(\mathrm{q}) \in R^{m\times k}$, $\mathrm{rank}(\mathbf{B}(\mathrm{q})) = m$ said the Jacobi matrix of constraint equation and the$\mathrm{D}\boldsymbol{\Phi}(\mathrm{q}) \cdot (\mathrm{DL}_q(\mathrm{e}) \cdot \widetilde{\mathrm{w}}) = \mathbf{B}(\mathbf{q})\mathbf{w}, (\mathrm{w} \in R^k)$, Where $\mathrm{D}\boldsymbol{\Phi}(\mathrm{q}) \cdot (\mathrm{DL}_q(\mathrm{e}) \cdot \widetilde{\mathrm{w}})$ is used to express the derivative of the constraint equation $\boldsymbol{\Phi}: \mathrm{G} \to R^m$. Thus, the structural equations, equilibria and constraints form the following equations of motion:

$$\begin{cases} \dot{\boldsymbol{q}} = DL_q(e) \cdot \tilde{V} \\ \boldsymbol{M}(q)\dot{V} = \boldsymbol{F}(q, V, t) - \boldsymbol{B^T}(q)\boldsymbol{\lambda} \\ \boldsymbol{\Phi}(q) = 0 \end{cases} \tag{19}$$

Formula (19) forms a system of index 3 differential-algebraic equations (DAEs) on a Lie group G. Where, the generalized mass matrix $\mathbf{M}(\mathrm{q})$ is a positive definite symmetric matrix composed of rigid body mass and moment of inertia, and $\mathbf{f}(\mathrm{q}, \mathrm{V}, t)$ is a generalized force vector in the form of column vectors. $\boldsymbol{\Phi}(\mathrm{q}) = 0$ is the complete constraint equation, $\lambda$ is the Lagrange multiplier, and $\mathbf{B}(\mathrm{q})$ is the Jacobian matrix of the constraint equation.

The constraint equations of velocity and acceleration can be obtained from the position constraint equations

$$\dot{\Phi}(\mathrm{q}) = \mathbf{B}(\mathrm{q})\mathrm{V} \tag{20}$$

$$\ddot{\Phi}(\mathrm{q}) = \dot{\mathbf{B}}(\mathrm{q})\mathrm{V} + \mathbf{B}(\mathrm{q})\dot{\mathrm{V}} = 0 \tag{21}$$

Using equation (17) and equation (18), the differential-algebraic systems of index 2 and index 1 on the Lie group G are, respectively

$$\begin{cases} \dot{\mathbf{q}} = D\mathrm{L}_q(\mathrm{e}) \cdot \widetilde{\mathrm{V}} \\ \mathbf{M}(\mathrm{q})\dot{\mathrm{V}} = -\mathrm{f}(\mathrm{q}, \mathrm{V}, t) - \mathrm{B}^T(\mathrm{q})\lambda \\ \mathbf{B}(\mathrm{q})\mathrm{V} = 0 \end{cases} \tag{22}$$

$$\begin{cases} \dot{\mathbf{q}} = D\mathrm{L}_q(\mathrm{e}) \cdot \widetilde{\mathrm{V}} \\ \mathbf{M}(\mathrm{q})\dot{\mathrm{V}} = -\mathbf{f}(\mathrm{q}, \mathrm{V}, t) - \mathbf{B^T}(\mathrm{q})\lambda \\ \dot{\mathbf{B}}(\mathrm{q})\mathrm{V} + \mathbf{B}(\mathrm{q})\dot{\mathrm{V}} = 0 \end{cases} \tag{23}$$

The equations (19), (22) and(23) contain only constraints on position, velocity and acceleration, respectively. In order to satisfy all constraints, this paper designs a constrained stability method for the motion model of Lie groups. Firstly, the equations (19)are improved, by reducing the index, introducing a new Lagrange multiplier μ, correcting the velocity and acceleration terms, forcing the velocity level constraints and acceleration level constraints to remain stable in the calculation process, and obtaining a stable index 1 differential algebraic equations

$$\begin{cases} \dot{\boldsymbol{q}} = DL_q(e) \cdot \widetilde{\Delta q} \\ \Delta \boldsymbol{q} = V - B^T(q)\mu \\ \Delta \boldsymbol{V} = \dot{V} - B^T(q)\omega \\ \boldsymbol{M}(q)\Delta V = F(q, V, t) - B^T(q)\lambda \\ \boldsymbol{\Phi}(q) = 0 \\ \boldsymbol{B}(q)V = 0 \\ \dot{\boldsymbol{B}}(q)V + \boldsymbol{B}(q)\dot{V} = 0 \end{cases} \tag{24}$$

Equation Group (24)is a stable index 1 differential algebraic system containing constraint equations about displacement, velocity and acceleration by using the constrained stability method.

In this section, we first introduce the relevant background content of Lie group theory in basic mathematics, including the exponential mapping which plays an important role in the subsequent calculation. Based on the theory of Lie group correlation, Lie group models of differential-algebraic equations of index 1,2,3 with developable structure are established and the constrained stability method is introduced, that is, the stable differential-algebraic equations of index 1 expressed by Lie group are constructed by reducing index 3 on the basis of index 3, which provides a computational model for the following dynamic simulation.

## 4. Numerical calculation of spatially deployable structures

A model with 3 developable elements is taken as an example for simulation experiment. Given the initial value, bar cross section radius $s = 0.03$, bar density $ro = 30$, transverse bar length $L1 = 2$, vertical bar length $L2 = 1$, and time step $h = 0.001$, Runge-Kutta is used to solve the differential-algebraic equation of Lie groups with developable structure. The solution results are shown in the figure below:

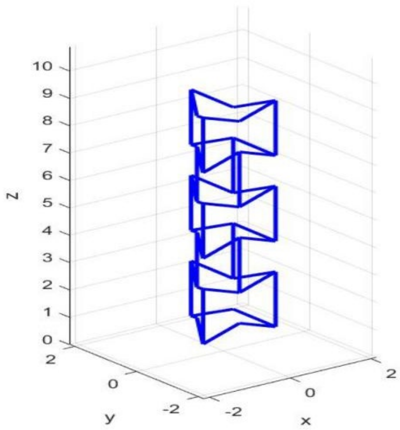


**Figure 2. Multideployable structure expansion diagram**

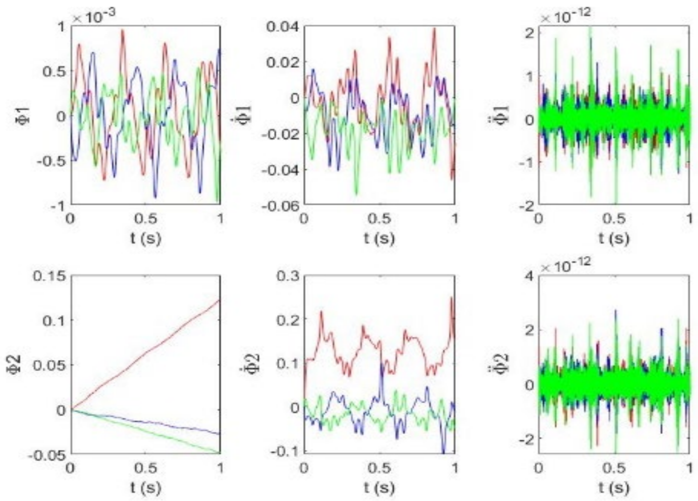


**Figure 3. Spatial developable structure model constraint change diagram**

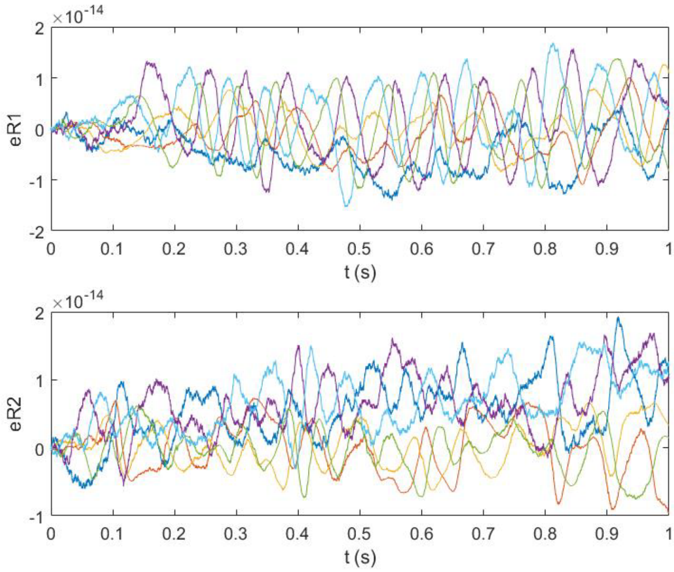


**Figure 4. Error variation diagram of rotation matrix orthogonality of developable structure model**

Figure 2 draws a model diagram of a multilayer deployable structure. Figure 3 shows the constraints at all levels of the two rods in the local diagram of the developable structure in Figure 1. It can be seen that the acceleration level constraints can be accurately maintained, while other constraints cannot be accurately maintained. Figure 4 shows the time variation of the orthogonality error $eR = R^T R - I_3$ of the same two-bar rotation matrix, that is, the structure of the Lie group in Runge-Kutta can also be preserved.

## 5. Conclusion

In this paper, numerical simulation of spatially deployable structures based on Lie group representation is carried out, index 1 differential-algebraic equations are established on Lie groups, and the Lie group format for the deployable structure model is designed using the classical numerical method explicit and implicit Runge-Kutta scheme, and the numerical simulation is carried out using MATLAB. The results show that the Lie group method can describe the attitude of the object by using special orthogonal group to avoid the strange phenomenon caused by parametric expression, and can satisfy the constraint conditions better, and keep the inherent Lie group property of the attitude of the object in the calculation process.

## Conflict and interest

The authors confirm that this article content has no conflict of interest.

## Acknowledgements

This study has been supported by the National Natural Science Foundations of China under Grant Nos. 12172186.

## References

[1] Y.N. Li, B.W. Li, J.Y. Ding, Z.K. Pan. Constrained Stabilization Methods for Differential Algebraic Equations of Lie Groups in Dynamics of Multi-body Systems. Chinese Journal of Dynamics and Control, 16 (2018) 97-101.

[2] Y.N. Li. Dynamic Simulation of Multi-body Systems Based on Lie Groups. Shandong: Qingdao University, 2019.

[3] Z.H. Huang, J. Chen, Z.J. Zhang, et al. Lie Group Variational integral Algorithm for multi-body Dynamics Simulation. Journal of Dynamics and Control, 20 (2022) 8-17.

[4] J. Chen, Z.H. Huang, Q. Tian. Lie Group Variational Integral Algorithm and Symmetric Reduction Method for multi-body Dynamics Simulation. The 18th Annual National Nonlinear Vibration and the 15th National Conference on Nonlinear Dynamics and Movement Stability (NVND2021). 2021.

[5] Y. Wang, Q.S. Lu, C. Liu. Geometric Properties of Euler Angle of Fixed Point Rotation of Rigid Body. Journal of Liaoning University (Natural Science Edition), 36 (2009) 197-200.

[6] L. Bai, Z.F. Dong, X.S. Ge. Underwater Navigation Based on Lie Group Dynamics Modeling and Optimal Control. Journal of System Simulation, 28 (2016) 1150-1157.

[7] J.Y. Ding, Z.K. Pan. The Lie Group Euler Methods of Multibody System Dynamics with Holonomic Constraints. Advances in Mechanical Engineering, 10 (2018) 1-10.

[8] J.M. Fon-Llagunes. Multibody Dynamics -Computational Methods and Applications. Springer, 2016.

[9] P. Betsch. Structure-preserving Integrators in Nonlinear Structural Dynamics and Flexible Multibody Dynamics.

Springer, 2016.

[10] P. Betsch, S. Uhlar. Energy-momentum conserving integration of multibody dynamics. Multibody System Dynamics, 17 (2007) 243-289.

[11] A. K. Mampetta. A Lie Group Formulation of Kinematics and Dynamics of Serial Manipulation. Mechhanics of Manipulation. (2006) 16-741.