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Existence for a class of evolutionary equations involving bilinear operators with applications

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Abstract: **In the paper we establish the existence of solutions to a class of evolutionary equations involving bilinear operators by Rothe's method. Our result is applied to the Navier-Stokes-Voigt equations.**

Keywords: Bilinear evolutionary equation; Rothe's method; Existence

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1. Introduction

Consider an evolution triple of spaces (V, H, V^*), that is, V is a separable reflexive Banach space, H is a separable Hilbert space, $V \subset H$ is continuous and V is dense in H. Let $\|\cdot\|_V$, $\|\cdot\|_H$ and $\|\cdot\|_{V^*}$ be the norms of V, H and V^*, respectively. Hence, we may regard that $V \subset H \equiv H^* \subset V^*$ and

$$\|x\|_{V^*} \le \|x\|_H \le \|x\|_V, \qquad \forall x \in V.$$

Moreover, we assume $V \subset H$ is compact. For given $0 < T < +\infty$, write

$$\mathsf{V} = L^2(0,T;V), \quad \mathsf{H} = L^2(0,T;H) \quad \text{and} \quad \mathsf{V}^* = L^2(0,T;V^*).$$

By the reflexivity of V, we easily check that V and V^* are reflexive Banach spaces. In this paper, we investigate the evolutionary equations:

$$\begin{cases} Ax'(t) + Bx(t) + C(x(t), x(t)) = f(t), & t \in (0,T), \\ x(0) = x_0, \end{cases} \tag{1.1}$$

where operators $A, B : V \to V^*$, $C : V \times V \to V^*$ and elements $f \in V^*$, $x_0 \in V$.

The evolutionary equations have been a cornerstone of scientific inquiry for centuries, tracing their roots back to the earliest days of modern mathematics. Many natural laws and complex phenomena can be described by the model of evolutionary equations (see [1, 2, 3]). In order to apply the more mature theory of ordinary differential equations to solve the corresponding problems in the evolutionary partial differential equations, people usually select appropriate state spaces to transform them into abstract differential equations (see [4, 5, 6, 7, 8]).

Recently, scholars have widely studied the existence of solutions to evolutionary equations. In [3], Wang and Fečkan investigated the existence of solutions for a general class of impulsive differential evolutionary equations. Liu et al. in [9] using KKM theorem studied the existence of solutions for an evolutionary equation mixing variational inequality. Zeng in [6] and [7] established existence results of solutions for a class nonlinear evolutionary equations involving weakly continuous operators and a class of time-fractional evolutionary equations, respectively.

The aim of this paper is to construct the existence result of abstract evolutionary equations involving bilinear operators. The rest of this paper is organized as follows. In Section 2, we will recall some preliminary nature and present some assumptions. In Section 3, we present the main result of this paper. In the last section, we apply the result to Navier-Stokes-Voigt equations.

2. Preliminaries

Let $\langle \cdot, \cdot \rangle_{X^*,X}$ be the duality pairing between X^* and X. The inner product of a Hilbert space Y will be denote by $\langle \cdot, \cdot \rangle_Y$. For $0 < T < +\infty$, let $L^p(0,T;X)$ denote the Banach space of p-integrable functions from $[0,T]$ to X with the norm

$$\|x\|_{L^p(0,T;X)} = \left(\int_0^T \|x(t)\|_X^p \, dt \right)^{\frac{1}{p}}, \quad 1 \le p < \infty, \quad \|x\|_{L^\infty(0,T;X)} = \operatorname*{ess\,sup}_{t \in [0,T]} \|x(t)\|_X$$

and $C([0,T];X)$ denote the Banach space of all continuous functions from $[0,T]$ to X with the norm $\|x\|_{C([0,T],X)} = \max_{[0,T]} \|x(t)\|_X$. The strong convergence and the weak convergence will be denoted by $\to$ and $\rightharpoonup$, respectively.

In order to deal with the time derivative in the evolutionary equation, we introduce the common spaces of all functions x whose time weak derivatives x' exists

$$\mathsf{W} = W^{1,2}(0,T;V) = \{x \in \mathsf{V} : x' \in \mathsf{V}\},$$

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endowed with the norm

$$\|x\|_{\mathsf{W}} = \|x\|_{\mathsf{V}} + \|x'\|_{\mathsf{V}}.$$

We now give the definition of solutions to (1.1).

Definition 2.1. A function $x \in \mathsf{W}$ is called a solution to (1.1) if it satisfies the following equations

$$\begin{cases} Ax'(t) + Bx(t) + C(x(t), x(t)) = f(t) \quad \text{in } V^* \text{ for a.e. } t \in (0,T), \\ x(0) = x_0. \end{cases}$$

We will review the definitions and results of single-valued operators as follows:

Definition 2.2 ([4, 10])**.** Let X, Y be Banach spaces. An operator $F : X \to Y$ is said to be weakly continuous, if for any sequence $\{u_n\} \subset X$ with $u_n \rightharpoonup u$ in X, we have $Fu_n \rightharpoonup Fu$ in Y.

Definition 2.3 ([4, 6])**.** Let X be a Banach space. An operator $F : X \to X^*$ is said to be coercive, if

$$\lim_{\|u\|_X \to \infty} \frac{\langle Fu, u \rangle_{X^*,X}}{\|u\|_X} = \infty.$$

The following fundamental surjectivity will be applied to establish the existence of solutions for evolutionary equations.

Theorem 2.4 ([4, 6])**.** Let X be a separable reflexive Banach space and $F : X \to X^*$ be weakly continuous and coercive. Then F is surjective, i.e., for every $f \in X^*$, there is $u \in X^*$ such that $Fu = f$.

We will need the following assumptions:

(H_A): $A : V \to V^*$ is linear bounded and there exists $\alpha > 0$ such that

$$\langle Ax, x \rangle_{V^*,V} \geq \alpha \|x\|_V^2, \qquad \forall x \in V.$$

(H_B): $B : V \to V^*$ is weakly continuous and there exist $\beta_1 > 0$, $\beta_2, \beta_3 \geq 0$ such that

$$\langle Bx, x \rangle_{V^*,V} \geq \beta_1 \|x\|_V^2, \quad \|Bx\|_{V^*} \leq \beta_1 + \beta_2 \|x\|_V, \qquad \forall x \in V.$$

(H_C): $C : V \times V \to V^*$ is bilinear and there exist $\gamma \geq 0$, $1 \geq \theta > 0$ such that

$$\langle C(x,x), x \rangle_{V^*,V} \geq 0, \quad \|C(x,y)\|_{V^*} \leq \gamma \|x\|_H^{\theta} \|x\|_V^{1-\theta} \|y\|_V, \qquad \forall x, y \in V.$$

3. Existence result

Theorem 3.1. If $(H_A), (H_B), (H_C), (H_f)$ hold, then (1.1) exists a solution.

We begin the proof of Theorem 3.1 with the following auxiliary problem.

Problem 3.2. Fix $x_0 \in V$, find $y \in \mathsf{W}$ such that

$$Ay(t) + B\Big(x_0 + \int_0^t y(s)\,ds\Big) + C\Big(x_0 + \int_0^t y(s)\,ds, x_0 + \int_0^t y(s)\,ds\Big) = f(t) \quad \text{in } V^* \text{ for a.e. } t \in (0,T).$$

Let $y(t) = x'(t)$ for a.e. $t \in (0,T)$. Then

$$x(t) = x_0 + \int_0^t y(s)\,ds, \qquad \forall t \in [0,T].$$

Obviously, $y \in \mathsf{V}$ solves Problem 3.2 if and only if $x \in \mathsf{W}$ is a solution of (1.1). Therefore, to prove Theorem 3.1 in what follows, it suffices to show that the existence of solutions to Problem 3.2.

We show the solvability of Problem 3.2 by applying Rothe's method. Let $N \in \mathbb{N}$ be fixed, $\tau = T/N$ denote the time step, and $t_k = k\tau$ for all $k = 1, \ldots, N$. For $v \in \mathsf{V}$, we define its Clément zero-order quasi-interpolant ([10, Remark 8.15]) by

$$v_\tau^k = \frac{1}{\tau} \int_{t_{k-1}}^{t_k} v(s)\,ds, \qquad \forall k = 1, \ldots, N.$$

For sufficiently small τ, we consider the approximation of the integral $\int_0^t v(s)\,ds$ by

$$\int_0^{t_n} v(s)\,ds = \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} v(s)\,ds = \tau \sum_{i=1}^{n} v_\tau^i.$$

We now consider the following discretized problem called Rothe's problem corresponding to Problem 3.2.

Problem 3.3. Find $\{y_\tau^k\}_{k=1}^N \subset V$ such that

$$Ay_\tau^k + Bz_\tau^k + C(z_\tau^k, z_\tau^k) = f_\tau^k, \qquad \forall k = 1, \ldots, N,$$

where $z_\tau^k \in V$ is given by

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$$z_\tau^k = x_0 + \tau\sum_{i=1}^{k} y_\tau^i.$$

We have the following existence result of Problem 3.3.

Lemma 3.4. Under assumptions of Theorem 3.1, then there is $\overline{\tau} > 0$ such that for any $\tau \in (0, \overline{\tau})$, Problem 3.3 has a solution $\{y_\tau^k\}_{k=1}^N \subset V$.

We note that Problem 3.3 can be equivalently formulated as follows: find $\{y_\tau^k\}_{k=1}^N \subset V$ such that

$$F(y_\tau^k) = f_\tau^k, \qquad \forall\, k = 1,\ldots,N, \tag{3.1}$$

where $F: V \to V^*$ is defined by

$$F(v) = Av + B\Big(x_0 + \tau\sum_{i=1}^{k-1} y_\tau^i + \tau v\Big) + C\Big(x_0 + \tau\sum_{i=1}^{k-1} y_\tau^i + \tau v, x_0 + \tau\sum_{i=1}^{k-1} y_\tau^i + \tau v\Big). \tag{3.2}$$

We will prove the solvability of (3.1) for small values of τ by showing that F is surjective. By Theorem 2.4, it is enough to show that F is weakly continuous and coercive. From assumptions $(H_A), (H_B)$ and (H_C), we easily check that F is weakly continuous. We now prove that F is coercive. Let

$$w_\tau = x_0 + \tau\sum_{i=1}^{k-1} y_\tau^i \quad \text{and} \quad M_\tau = \|x_0\|_V + \tau\sum_{i=1}^{k-1}\|y_\tau^i\|_V.$$

For $v \in V$, we set

$$v_\tau = w_\tau + \tau v.$$

Hence,

$$\|v_\tau\|_V = \|w_\tau + \tau v\|_V \le \|w_\tau\|_V + \tau\|v\|_V \le M_\tau + \tau\|v\|_V.$$

Multiplying (3.2) by $v \in V$, we have

$$\begin{aligned}
&\langle Fv, v\rangle_{V^*,V}\\
&= \langle Av + Bv_\tau + C(v_\tau, v_\tau), v\rangle_{V^*,V}\\
&= \langle Av, v\rangle_{V^*,V} + \frac{1}{\tau}\langle Bv_\tau, v_\tau\rangle_{V^*,V} - \frac{1}{\tau}\langle Bv_\tau, w_\tau\rangle_{V^*,V} + \frac{1}{\tau}\langle C(v_\tau, v_\tau), v_\tau\rangle_{V^*,V} - \frac{1}{\tau}\langle C(v_\tau, v_\tau), w_\tau\rangle_{V^*,V}\\
&\ge \alpha\|v\|_V^2 + \frac{\beta_1}{\tau}\|v_\tau\|_V^2 - \frac{1}{\tau}[\beta_2 + \beta_3(M_\tau + \tau\|v\|_V)]M_\tau - \frac{\gamma}{\tau}(M_\tau + \tau\|v\|_V)^2 M_\tau\\
&\ge (\alpha - 2\tau\gamma M_\tau)\|v\|_V^2 + \frac{\beta_1}{\tau}\|v_\tau\|_V^2 - \beta_3 M_\tau\|v\|_V - \frac{\beta_2 M_\tau + \beta_3 M_\tau + 2\gamma M_\tau^3}{\tau}.
\end{aligned} \tag{3.3}$$

Choose τ such that $\alpha - 2\tau\gamma M_\tau \ge \alpha/2$. Then, we deduce that F is coercive for all $\tau \in (0, \overline{\tau})$.

Lemma 3.5. Under assumptions of Theorem 3.1, then there is $\overline{\tau} > 0$ such that for any $\tau \in (0, \overline{\tau})$, the following estimates are satisfied

$$\max_{k=1,\ldots,N}\|y_\tau^k\|_V \le M, \tag{3.4}$$

$$\max_{k=1,\ldots,N}\|z_\tau^k\|_V \le M, \tag{3.5}$$

where $M > 0$ is independent of τ.

Proof. For every $1 \le k \le N$, we have

$$Ay_\tau^k + Bz_\tau^k + C(z_\tau^k, z_\tau^k) = f_\tau^k. \tag{3.6}$$

Choose τ such that $\alpha - 2\overline{\tau}\gamma M_\tau \ge \alpha/2$. We now consider following two cases for $\tau \in (0, \overline{\tau})$.

Case 1. For $\beta_2\|x_0\|_V + \beta_3\|x_0\|_V + \gamma\|x_0\|_V^3 \ge \beta_1\|z_\tau^k\|_V^2$, we have $\{z_\tau^k\}_{k=1}^N$ is bounded in V. Due to $f \in C([0,T]; V^*)$, there is a constant $M_f > 0$ such that $\|f_\tau^k\|_{V^*} \le M_f$ for all $\tau > 0$ and $k \in \mathbb{N}$. Multiplying (3.6) by y_τ^k, we obtain

$$\begin{aligned}
\alpha\|y_\tau^k\|_V^2 &\le \langle Ay_\tau^k, y_\tau^k\rangle_{V^*,V}\\
&= \langle f_\tau^k - Bz_\tau^k - C(z_\tau^k, z_\tau^k), y_\tau^k\rangle_{V^*,V}\\
&\le \|f_\tau^k\|_{V^*}\|y_\tau^k\|_V + \|Bz_\tau^k\|_{V^*}\|y_\tau^k\|_V + \|C(z_\tau^k, z_\tau^k)\|_{V^*}\|y_\tau^k\|_V\\
&\le (M_f + \beta_1 + \beta_2\|z_\tau^k\|_V + \gamma\|z_\tau^k\|_V^2)\|y_\tau^k\|_V,
\end{aligned}$$

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or

$$\alpha \left\| y_\tau^k \right\|_V \le M_f + \beta_1 + \beta_2 \left\| z_\tau^k \right\|_V + \gamma \left\| z_\tau^k \right\|_V^2,$$

which implies that $\{y_\tau^k\}_{k=1}^N$ is bounded in V.

Case 2. For $\beta_2 \left\| x_0 \right\|_V + \beta_3 \left\| x_0 \right\|_V + \gamma \left\| x_0 \right\|_V^3 < \beta_1 \left\| z_\tau^k \right\|_V^2$, multiplying (3.6) by y_τ^k and using the similar arguments of (3.3), we have

$$\begin{aligned} \langle f_\tau^k, y_\tau^k \rangle_{V^*,V} &= \langle A y_\tau^k + B z_\tau^k + C(z_\tau^k, z_\tau^k), y_\tau^k \rangle_{V^*,V} \\ &\ge (\alpha - 2\tau\gamma M_\tau) \left\| y_\tau^k \right\|_V^2 + \frac{\beta_1}{\tau} \left\| z_\tau^k \right\|_V^2 - \beta_3 M_\tau \left\| y_\tau^k \right\|_V - \frac{\beta_2 M_\tau + \beta_3 M_\tau + 2\gamma M_\tau^3}{\tau}. \end{aligned} \tag{3.7}$$

If follows from (3.7) that

$$\begin{aligned} \left\| y_\tau^k \right\|_V^2 &\le \frac{2}{\alpha}\left(\beta_3 M_\tau + M_f\right) \left\| y_\tau^k \right\|_V + \frac{2}{\alpha\tau}(\beta_2 M_\tau + \beta_3 M_\tau + 2\gamma M_\tau^3) - \frac{2\beta_1}{\alpha\tau} \left\| z_\tau^k \right\|_V^2 \\ &= \frac{2}{\alpha}\left(\beta_3 M_\tau + M_f\right) \left\| y_\tau^k \right\|_V + \frac{2(\beta_2 + \beta_3)}{\alpha\tau}\left(\left\| x_0 \right\|_V + \tau \sum_{i=1}^{k-1} \left\| y_\tau^i \right\|_V \right) + \frac{2\gamma}{\alpha\tau}\left(\left\| x_0 \right\|_V + \tau \sum_{i=1}^{k-1} \left\| y_\tau^i \right\|_V \right)^3 - \frac{2\beta_1}{\alpha\tau} \left\| z_\tau^k \right\|_V^2 \\ &= \frac{2}{\alpha}\left(\beta_3 M_\tau + M_f\right) \left\| y_\tau^k \right\|_V + \frac{2}{\alpha\tau}(\beta_2 \left\| x_0 \right\|_V + \beta_3 \left\| x_0 \right\|_V + \gamma \left\| x_0 \right\|_V^3 - \beta_1 \left\| z_\tau^k \right\|_V^2) + \frac{2}{\alpha}(\beta_2 + \beta_3 + 3\gamma \left\| x_0 \right\|_V) \sum_{i=1}^{k-1} \left\| y_\tau^i \right\|_V \\ &\quad + \frac{6\gamma\tau}{\alpha} \left\| x_0 \right\|_V \left(\sum_{i=1}^{k-1} \left\| y_\tau^i \right\|_V \right)^2 + \frac{2\gamma\tau^2}{\alpha} \left(\sum_{i=1}^{k-1} \left\| y_\tau^i \right\|_V \right)^3 \\ &\le \frac{2}{\alpha}\left(\beta_3 M_\tau + M_f\right) \left\| y_\tau^k \right\|_V + \frac{2}{\alpha}(\beta_2 + \beta_3 + 3\gamma \left\| x_0 \right\|_V) \sum_{i=1}^{k-1} \left\| y_\tau^i \right\|_V + \frac{6\gamma\tau}{\alpha} \left\| x_0 \right\|_V \left(\sum_{i=1}^{k-1} \left\| y_\tau^i \right\|_V \right)^2 + \frac{2\gamma\tau^2}{\alpha} \left(\sum_{i=1}^{k-1} \left\| y_\tau^i \right\|_V \right)^3. \end{aligned} \tag{3.8}$$

Apply the elementary fact that if $x^2 \le ax + b$ for $a, b, x \ge 0$, then $x \le a + \sqrt{b}$. Hence, (3.8) shows that

$$\left\| y_\tau^k \right\|_V \le \frac{2}{\alpha}(\beta_3 M_\tau + M_f) + \left(\frac{2}{\alpha}(\beta_2 + \beta_3 + 3\gamma \left\| x_0 \right\|_V) \sum_{i=1}^{k-1} \left\| y_\tau^i \right\|_V + \frac{6\gamma\tau}{\alpha} \left\| x_0 \right\|_V \left(\sum_{i=1}^{k-1} \left\| y_\tau^i \right\|_V \right)^2 + \frac{2\gamma\tau^2}{\alpha} \left(\sum_{i=1}^{k-1} \left\| y_\tau^i \right\|_V \right)^3 \right)^{\frac{1}{2}}. \tag{3.9}$$

Clearly,

$$\left\| y_\tau^1 \right\|_V \le \frac{2}{\alpha}(\beta_3 M_\tau + M_f),$$

which yields that y_τ^1 is bounded in V. From this and (3.9), we can deduce that $\{y_\tau^k\}_{k=1}^N$ is bounded in V by induction. Moreover, note that

$$\left\| z_\tau^k \right\|_V = \left\| x_0 + \tau \sum_{i=1}^{k} y_\tau^i \right\|_V \le \left\| x_0 \right\|_V + \tau \sum_{i=1}^{k} \left\| y_\tau^i \right\|_V.$$

Thus, we obtain $\{z_\tau^k\}_{k=1}^N$ is bounded in V by the boundedness of $\{y_\tau^k\}_{k=1}^N$. □

In the final step of Rothe's method, we show the convergence of appropriately defined approximate sequences of functions towards a solution to Problem 3.2. Consider the piecewise constant interpolants $\tilde{y}_\tau, \tilde{z}_\tau \in \mathsf{V}$ and $\tilde{f}_\tau \in \mathsf{V}^*$ that are given by

$$\tilde{y}_\tau(t) = y_\tau^k, \qquad \tilde{z}_\tau(t) = z_\tau^k, \qquad \tilde{f}_\tau(t) = f_\tau^k$$

for a.e. $t \in (t_{k-1}, t_k]$ with $k = 1, \ldots, N$.

Lemma 3.6. Under assumptions of Theorem 3.1, then for a subsequence, we have

$$\tilde{y}_\tau \rightharpoonup y \quad \text{in } \mathsf{V}, \quad \text{as} \quad \tau \to 0,$$

where $y \in \mathsf{V}$ is the solution of Problem 3.2.

Proof. From (3.4), we see that $\{\tilde{y}_\tau\}$ is bounded in V. Hence, there is a subsequence of $\{\tilde{y}_\tau\}$, denoted in the same way, such that

$$\tilde{y}_\tau \rightharpoonup y \quad \text{in } \mathsf{V}, \quad \text{as} \quad \tau \to 0. \tag{3.10}$$

Let $\Phi : \mathsf{V} \to C([0,T]; V)$ be defined by

$$(\Phi v)(t) = \int_0^t v(s)\, ds, \qquad \forall t \in [0,T].$$

Clearly, Φ is weakly continuous. This leads to $\Phi\tilde{y}_\tau \rightharpoonup \Phi y$ in $C([0,T]; V)$ as $\tau \to 0$. Hence,

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$$\Phi\tilde{y}_\tau(t) \rightharpoonup \Phi y(t) \quad \text{in } V, \quad \text{as} \quad \tau \to 0 \tag{3.11}$$

for all $t \in [0,T]$. From (3.4), there is a constant $M > 0$ such that $\|\tilde{y}_\tau(s)\|_V \le M$ for a.e. $s \in (0,T)$. Then, for $t \in (t_{k-1}, t_k]$, we have

$$\|\tilde{z}_\tau(t) - x_0 - \Phi\tilde{y}_\tau(t)\|_V = \left\|\int_0^{t_k} \tilde{y}_\tau(s)\,ds - \int_0^t \tilde{y}_\tau(s)\,ds\right\|_V \le \int_t^{t_k} \|\tilde{y}_\tau(s)\|_V\,ds \le M\,|t_k - t|,$$

which means that

$$\tilde{z}_\tau(t) - x_0 - \Phi\tilde{y}_\tau(t) \to 0 \quad \text{in } V, \quad \text{as} \quad \tau \to 0 \tag{3.12}$$

for all $t \in [0,T]$. Combining (3.11) with (3.12), we have

$$\tilde{z}_\tau(t) \rightharpoonup x_0 + \Phi y(t) \quad \text{in } V, \quad \text{as} \quad \tau \to 0 \tag{3.13}$$

for all t ∈ [0,T]. Consider the Nemytski operators $\mathsf{A} : \mathsf{V} \to \mathsf{V}^*$, $\mathsf{B} : \mathsf{V} \to \mathsf{V}^*$, $\mathsf{C} : \mathsf{V} \times \mathsf{V} \to \mathsf{V}^*$ corresponding to A, B, C, respectively, i.e.,

$$(\mathsf{A}v)(t) = Av(t), \quad (\mathsf{B}v)(t) = Bv(t), \quad (\mathsf{C}(v,w))(t) = C(v(t), w(t))$$

for a.e. $t \in (0,T)$. Then (3.6) shows that

$$\mathsf{A}\tilde{y}_\tau + \mathsf{B}\tilde{z}_\tau + \mathsf{C}(\tilde{z}_\tau, \tilde{z}_\tau) = \tilde{f}_\tau. \tag{3.14}$$

By hypothesis (H_A) and (3.10), we have

$$\mathsf{A}\tilde{y}_\tau \rightharpoonup \mathsf{A}y \quad \text{in } \mathsf{V}^*, \quad \text{as} \quad \tau \to 0.$$

From (3.13) and Lebesgue's dominated convergence theorem [11, Theorem 1.65], we have

$$\begin{aligned}\lim_{n\to\infty}\langle \mathsf{B}\tilde{z}_\tau, v\rangle_{\mathsf{V}^*,\mathsf{V}} &= \lim_{n\to\infty}\int_0^T \langle B\tilde{z}_\tau(t), v(t)\rangle_{V^*,V}\,dt = \int_0^T \lim_{n\to\infty}\langle B\tilde{z}_\tau(t), v(t)\rangle_{V^*,V}\,dt \\ &= \int_0^T \langle B(x_0 + \Phi y(t)), v(t)\rangle_{V^*,V}\,dt = \langle \mathsf{B}(x_0 + \Phi y), v\rangle_{\mathsf{V}^*,\mathsf{V}}\end{aligned}$$

for all $v \in \mathsf{V}$, which yields that $\mathsf{B}\tilde{z}_\tau \rightharpoonup \mathsf{B}(x_0 + \Phi y)$ in V^*, as $\tau \to 0$. Clearly,

$$\begin{aligned}&\langle \mathsf{C}(\tilde{z}_\tau, \tilde{z}_\tau), v\rangle_{\mathsf{V}^*,\mathsf{V}} - \langle \mathsf{C}(x_0 + \Phi y, x_0 + \Phi y), v\rangle_{\mathsf{V}^*,\mathsf{V}} \\ &= \int_0^T \langle C(\tilde{z}_\tau(t), \tilde{z}_\tau(t)) - C(x_0 + \Phi y(t), x_0 + \Phi y(t)), v(t)\rangle_{V^*,V}\,dt \\ &= \int_0^T \langle C(\tilde{z}_\tau(t) - x_0 - \Phi y(t), \tilde{z}_\tau(t)), v(t)\rangle_{V^*,V}\,dt + \int_0^T \langle C(x_0 + \Phi y(t), \tilde{z}_\tau(t) - x_0 - \Phi y(t)), v(t)\rangle_{V^*,V}\,dt.\end{aligned} \tag{3.15}$$

By hypothesis (H_C) and (3.13), we have

$$\int_0^T \langle C(x_0 + \Phi y(t), \tilde{z}_\tau(t) - x_0 - \Phi y(t)), v(t)\rangle_{V^*,V}\,dt \to 0, \quad \text{as} \quad \tau \to 0. \tag{3.16}$$

Since V is compactly embedded in H, using (3.5) and (3.11), we get

$$\begin{aligned}&\int_0^T \langle C(\tilde{z}_\tau(t) - x_0 - \Phi y(t), \tilde{z}_\tau(t)), v(t)\rangle_{V^*,V}\,dt \\ &\le \int_0^T \gamma \|\tilde{z}_\tau(t) - x_0 - \Phi y(t)\|_H^{\theta} \|\tilde{z}_\tau(t) - x_0 - \Phi y(t)\|_V^{1-\theta} \|\tilde{z}_\tau(t)\|_V \|v(t)\|_V\,dt \\ &\le \max_{t\in[0,T]} \|\tilde{z}_\tau(t) - x_0 - \Phi y(t)\|_H^{\theta} \cdot \max_{t\in[0,T]} \|\tilde{z}_\tau(t) - x_0 - \Phi y(t)\|_V^{1-\theta} \gamma \|\tilde{z}_\tau\|_{\mathsf{V}} \|v\|_{\mathsf{V}} \to 0,\end{aligned}$$

as $\tau \to 0$. This, combining (3.15) with (3.16), we have

$$\mathsf{C}(\tilde{z}_\tau, \tilde{z}_\tau) \rightharpoonup \mathsf{C}(x_0 + \Phi y, x_0 + \Phi y) \quad \text{in } \mathsf{V}^*, \quad \text{as} \quad \tau \to 0.$$

It follows from ([1, Lemma 3.3]) that $\tilde{f}_\tau \to f$ in V^*, as $\tau \to 0$. Using above convergences and (3.14), we obtain

$$\mathsf{A}y + \mathsf{B}(x_0 + \Phi y) + \mathsf{C}(x_0 + \Phi y, x_0 + \Phi y) = f,$$

which implies that $y \in \mathsf{V}$ is a solution to Problem 3.2. □

Combining Lemma 3.4, Lemma 3.5 and Lemma 3.6, Theorem 3.1 is established.

4. Navier-Stokes-Voigt equations

Let Ω be an open bounded domain in $\mathbb{R}^3$ with C^1 boundary $\partial\Omega$. Given $0 < T < +\infty$, we set $I = \Omega \times (0,T)$. Consider the three-dimensional (3D) Navier-Stokes-Voigt equations:

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$$\begin{cases} z' - \nu\Delta z - \alpha^2\Delta z' + (z\cdot\nabla)z + \nabla p = f, & \text{in } I, \\ \nabla\cdot z = 0, & \text{in } I, \\ z(x,t) = 0, & \text{on } \partial\Omega\times(0,T), \\ z(x,0) = z_0(x), & \text{in } \Omega. \end{cases} \tag{4.1}$$

Here, $z = z(t,x) = (z_1(t,x), z_2(t,x), z_3(t,x))$ is the velocity vector field, p is the pressure, $\nu > 0$ is the kinematic viscosity, α is the length scale that characterizes the elasticity parameter of the fluid. Define

$$V_0 = \{z \in (C_0^\infty(\Omega))^3 : \nabla z = 0\}.$$

In this part, let V and H be the closure of V_0 in $(W_0^{1,2}(\Omega))^3$ and $(L^2(\Omega))^3$, respectively, with their inner products and norms

$$\langle z, w\rangle_V := \int_\Omega \sum_{j=1}^3 \nabla z_j \cdot \nabla w_j \, dx, \quad \|z\|_V = \sqrt{\langle z, z\rangle_V}, \qquad \forall z, w \in V,$$

$$\langle z, w\rangle_H := \int_\Omega \sum_{j=1}^3 z_j w_j \, dx, \quad \|z\|_H = \sqrt{\langle z, z\rangle_H}, \qquad \forall z, w \in H.$$

Obviously, V and H are separable Hilbert spaces. The embedding $V \subset H \equiv H^* \subset V^*$ are dense, continuous and compact. Hence, we may regard that

$$\|z\|_{V^*} \le \|z\|_H \le \|z\|_V, \qquad \forall z \in V.$$

We now define the trilinear form b by

$$b(y,z,w) := \sum_{i,j}^3 \int_\Omega y_i \frac{\partial z_j}{x_i} w_j \, dx,$$

which satisfies

$$b(y,z,w) = -b(y,w,z), \qquad b(y,z,z) = 0, \qquad \forall y, z, w \in V.$$

We call $z \in \mathsf{W}$ the weak solution of problem (4.1) if

$$\begin{cases} \langle z'(t), v\rangle_H + \nu\langle z(t), v\rangle_V + \alpha^2\langle z'(t), v\rangle_V + b(z(t), z(t), v) = \langle f(t), v\rangle_{V^*,V} & \text{for a.e. } t \in (0,T), \\ z(0) = z_0. \end{cases}$$

Theorem 4.1. Fix $z_0 \in V$ and $f \in \mathsf{V}^*$, (4.1) has a weak solution.

Proof. Let $\bar{A} : V \to V^*, \bar{B} : V \to V^*, \bar{C} : V \times V \to V^*$ be given by

$$\langle \bar{A}u, v\rangle_{V^*,V} = \langle u, v\rangle_H + \alpha^2\langle u, v\rangle_V, \quad \langle \bar{B}u, v\rangle_{V^*,V} = \nu\langle u, v\rangle_V, \quad \langle \bar{C}(u,v), w\rangle_{V^*,V} = b(u,v,w), \qquad \forall u, v, w \in V.$$

It is clear that $\bar{A}$ and $\bar{B}$ satisfy (H_A) and (H_B), respectively. For $v \in V$, we easily check that

$$\langle \bar{C}(v,v), v\rangle_{V^*,V} = b(v,v,v) = 0.$$

Moreover, we have (see [12])

$$\langle \bar{C}(u,v), w\rangle_{V^*,V} = b(u,v,w) \le M \|u\|_H^{\frac{1}{4}} \|u\|_V^{\frac{3}{4}} \|v\|_V \|w\|_V, \quad \forall u, v, w \in V.$$

These show that $\bar{C}$ satisfies (H_C). By Theorem 3.1, problem (4.1) has a weak solution. □

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