Journal of Computational Science & Engineering 62 (2023) 1-5

Available at http://www.asocse.org ISSN 1710-4068

A kind of nonsmooth Newton method for discrete stochastic absolute value equations

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Abstract: **In this paper, we propose a kind of stochastic absolute value equations. And a kind of nonsmooth Newton method is also proposed. Under mild assumption, the convergence analysis of the method is given. Finally, numerical results indicate the effeciency of the method.**

Keywords: stochastic absolute value equations; nonsmooth Newton method; convergence

Received 22 November 2023, Revised 7 November 2023, Accepted 10 November 2023

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1.Introduction

In the past few decades, various stochastic problems have been widely applied in solving many mathematical problems with uncertainty, among which the stochastic absolute value equations [1] with stochastic optimization characteristics is a class of nonsmooth stochastic optimization problems.

The problem of stochastic absolute value equations considered in this paper can be expressed as find a vector $x \in R^n$ such that

$$A(\omega)x - |x| = b(\omega), \quad (1)$$

where $A(\omega) \in R^{n\times n}$ and $b(\omega) \in R^n$ for $\omega \in \Omega$， where Ω is a kind of discrete probability space with finitely many elements, i.e.,$\Omega = \{\omega_1, \omega_2, \dots, \omega_n\}$, $p_i = P\{\omega_i \in \Omega\}$. $|x|$ is the component wise absolute value of vector $x \in R^n$.

Let (Ω, F, ρ) be a probability space, where Ω is a subset of R^n and ρ is a standard probability measure. Here, we assume that ρ is known. Let ω be a stochastic variable in the same sample. If Ω only contains one element, the stochastic absolute value equations problem can degenerate into an absolute value equation, and the absolute value equations can be transformed into a general linear complementarity problem for solution. The absolute value equations are widely used in solving location problems and mathematical programming problems [2-4]. Absolute value equations can be expressed as find a vector $x \in R^n$ such that

$$Ax - |x| = b, \quad (2)$$

where $A \in R^{n\times n}$ and $b \in R^n$， $|x|$ is the component wise absolute value of vector $x \in R^n$. Recently, some computational methods for solving absolute value equations have been proposed. In [5,6], the concave minimization formula is solved using the continuous linearization algorithm. However, it is unknown whether this method has global convergence. In [7], a generalized Newton method was proposed, which greatly reduces the computational time of the continuous linearization algorithm. It has been proven that the definition and boundedness of the generalized Newton iteration number are good when the singular value of A is greater than 1. In [8], the authors proposed another generalized Newton method that combines the generalized Newton step in [7] and the Jacobian matrix smoothing Newton step in [9]. In each iteration, one to two linear equations need to be solved. In [10], a smooth Newton method for absolute value equations was proposed. When the singular value of A is greater than 1, it is proved that the method has global convergence and the convergence rate is quadratic. In [11], a smooth Newton algorithm is used to solve absolute value equations under conditions where the singular value is greater than 1. (1) can be expressed as

$$\mathrm{H}(x) = A(\omega)x - |x| - b(\omega) = 0. \quad (3)$$

The above equation is a class of nonsmooth absolute value equations. There are already mature methods for general nonsmooth equation problems, such as the various nonsmooth methods mentioned in reference [3,12-13], which have good convergence properties. Whether this type of method can be extended to solve the problem of stochastic absolute value equations. This paper takes this as a motivation to consider the algorithm for solving the stochastic absolute value equations method.

In [1], considering the equivalence between stochastic bilinear programming and stochastic generalized linear complementarity problems in a general probability space Ω, the expected residual minimization method and the expected value method are applied to transform the stochastic absolute value equation problem into a deterministic problem for solution. So, in this paper, similar to the expected value method in [1] to solve (1). E[·]stands for the expectation of every elements of matrix and vector. $\bar{A}$ denotes the expectation of $A(\omega)$ and $\bar{b}$ denotes the expectation of $b(\omega)$,i.e.,

$$\bar{A}=\mathrm{E}[A(\omega)], \bar{b}=\mathrm{E}[b(\omega)].$$

So solving (1) can be transformed into

$$\bar{H}(x) = \bar{A}x - |x| - \bar{b}=0. \quad (4)$$

Next, we will study the nonsmooth Newton method for solving stochastic absolute value and provides relevant numerical experimental examples to demonstrate the effectiveness of the proposed method.

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2.Preliminaries

Let $F: R^n \to R^m$ be a local Lipschitz function. According to Rademacher's theorem, F is almost continuous everywhere. $D_F$is the differentiable point set of the function. The generalized Jacobian matrix of function F at point x is denoted as $\partial F(x)$

$$\partial F(x)=conv\left\{\lim_{\substack{x^k\to x\\ x^k\in D_F}} F'(x_k)\right\}.$$

Definition 1 $F: R^n \to R^m$ is a local Lipschitz function. The limit

$$\lim_{\substack{V\in\partial F(x+th')\\ h'\to h,t\downarrow 0}}\{Vh'\}$$

exists for any $h \in R^n$, it is called a semi-smooth function.

Proposition 1 If $\overline{H}$ is a function defined in (4), then $\overline{H}$is a semi-smooth function, and the generalized Jacobian matrix can be expressed as:

$$\partial\overline{H}(x)=\{\bar{A}-D(x)\},$$

where $D(x)=\partial|x|=diag(\partial|x_i|), i\in\{1,2,\dots,n\}$,

i.e.,

$$\partial|x_i|=\begin{cases}1 & x_i>0,\\ -1 & x_i<0,\\ \varsigma_i\in[-1,1] & x_i=0.\end{cases}$$

Proof Since $A(\omega)x$ is a smooth function about x, as known from [3], $|x|$is a semi smooth function, then F is a semi-smooth function, and for the subdifferential of $|x|$ is $D(x)$,$D(x)=\partial|x|=diag(\partial|x_i|)$, then the proposition holds.

Proposition 2[13] If F is a semi-smooth function, then

(i)when $\forall\, x\in F(x+h), h\to 0$, there is

$$V\in\partial F(x+h),$$

$$Vh-F'(x;h)=o(||h||),$$

(ii) $$\lim_{\substack{x+h\in D_F\\ h\to 0}}\frac{F(x+h)-F(x)-F'(x;h)}{||h||}=0.$$

Proposition 3[13] If all $V\in\partial F(x)$ are non-singular, then there exists a neighborhood $N(x)$of x and a constant C, such that for any pair of $y\in N(x)$, $V\in\partial F(y)$, V is non-singular and satisfies $||V^{-1}||\le C$.

Proposition 4[13]If F is semi-smooth at x, then $\forall\, h\to 0$, we have.

$$F(x+h)-F(x)-F'(x;h)=o(||h||).$$

3.Nonsmooth Newton method

This section provides a kind of nonsmooth Newton methods for solving problem (4) and provides convergence analysis of the algorithm.

Algorithm 1

Step 0 Select $x_0\in N(x^*,\delta)$, $0<\varepsilon<1, 0<\delta<1$, let $k=0$.

Step 1 If $\|\overline{H}(x_0)\|<\varepsilon$, stop; otherwise, calculate $V_k\in\partial\overline{H}(x_k)$.

Step 2 let

$$x_{k+1}=x_k-(V_k+\lambda_k I)^{-1}\overline{H}(x_k), \quad (5)$$

$k:=k+1$, Step 1.

In (5), I is the $n\times n$ identity matrix, $\partial\overline{H}(x_k)$ is taken by Proposition 1 and parameters λ_k are chosen to ensure $\{x_k\}$ convergence and $V_k+\lambda_k I$ is invertible. A simple strategy is to set $\lambda_k=0$ if V_k is non-singular and $\lambda_k=\|V_k\|+\varepsilon$ if V_k is singular, where ε is a small positive number. Similar to [14], the following convergence theorem is given.

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Theorem 1 Suppose that x^* is a solution of (4), and all $V^* \in \partial\bar{H}(x^*)$ are non-singular. Let ε and λ_k satisfy

$$0 < 3\beta\varepsilon < 1$$

and

$$|\lambda_k| \le \lambda' < \frac{1-3\beta\varepsilon}{2\beta}.$$

For any $x_0 \in N(x^*, \delta)$, $\delta > 0$, the sequence $\{x_k\}$ defined by (5) linearly converges to x^*. Furthermore, if $\lambda_k \to 0$ as $k \to \infty$, then $\{x_k\}$ convergence superlinearly to x^*.

Proof By Proposition 2, there exists a constant $\delta > 0$ such that for any $x \in N(x^*, \delta)$, and $V^x \in \partial\bar{H}(x)$, we have

$\|\bar{H}(x) - \bar{H}(x^*) - \bar{H}'(x^*; x - x^*)\| \le \varepsilon\|x - x^*\|$,

$\|V^x(x - x^*) - \bar{H}'(x^*; x - x^*)\| \le \varepsilon\|x - x^*\|$,

$\|V^x - V^*\| \le \varepsilon, V^* \in \partial\bar{H}(x^*)$.

Hence

$$\|V^x - \lambda I - V^*\| \le \varepsilon + |\lambda| < \frac{1}{\beta}.$$

If $|\lambda| \le \lambda'$, this shows that $V^x + \lambda I$ is non-singular and

$$\|(V^* + \lambda I)^{-1}\| \le \frac{\beta}{1 - \beta(\varepsilon + |\lambda|)}$$

So (5) is well-defined for $x_k \in N(x^*, \delta)$.

By $\|V^k\| \le \varepsilon + \|V^*\|$, we get

$\|x_{k+1} - x^*\| \le \|(V^k + \lambda_k I)^{-1}\|(\|\bar{H}(x_k) - \bar{H}(x^*) - \bar{H}'(x^*; x_k - x^*)\| + \|V^k(x_k - x^*) - \bar{H}'(x^*; x_k - x^*) + |\lambda_k|)\| \le \frac{\beta}{1-\beta(\varepsilon+|\lambda_k|)}(2\varepsilon + |\lambda_k|)\|x_k - x^*\| \le \rho\|x_k - x^*\|$,

where

$$\rho = \frac{\beta}{1-\beta(\varepsilon+|\lambda_k|)}(2\varepsilon + \lambda') < 1.$$

Hence $\{x_k\}$ convergence linearly to x^*.

By

$$\|\bar{H}(x) - \bar{H}(x^*) - \bar{H}'(x^*; x - x^*)\| = o(\|x - x^*\|),$$
$$\|V^x(x - x^*) - \bar{H}'(x^*; x - x^*)\| = o(\|x - x^*\|),$$

let $\lambda_k \to 0$ as $k \to \infty$, we have

$$\|x_{k+1} - x^*\| = o(\|x_k - x^*\|).$$

Hence $\{x_k\}$defined by (5) convergence superlinearly to x^*.

4.Numerical results

This section provides relevant numerical experiments to demonstrate the effectiveness of the given non smooth Newton methods for solving stochastic absolute value equations. The algorithm program is written in Matlab.

Example 1 Consider (1)，where $A(\omega)$ =

$$\begin{pmatrix} 2+\omega & 1 & & & & & \\ 1 & 2+\omega & 1 & & & & \\ & 1 & 2+\omega & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 1 & 2+\omega & 1 & \\ & & & & 1 & 2+\omega & \end{pmatrix}_{n\times n}$$

$b(\omega) = (2+\omega, 3+\omega, 3+\omega, \cdots, 3+\omega, 2+\omega)^T$,

$\Omega = \{\omega_1, \omega_2\} = \{0,1\}$，$p_i = P\{\omega_i \in \Omega\} = \frac{1}{2}$，$i = 1,2$.

According to reference [1], its exact solution is $(1,1,\ldots 1)^T$. Algorithm 1 is applied to the above Example 1, taking five sets of initial points within the δ neighborhood of the exact solution x, where x_0 is the initial point and $x^*$is the optimal value. The relevant numerical results are shown in Table 1. We take $\varepsilon = 10^{-6}$， δ=2, λ_k=0, n=100 or 200.

Table 1 Example 1 numerical results(n=100)

x_0	x^*	$F(x)$
$(0.5761, \ldots, 0.2076)^T$	$(1,1,\ldots 1)^T$	4.8648e-15
$(0.2239, \ldots, 0.8756)^T$	$(1,1,\ldots 1)^T$	4.2596e-15
$(0.6233, \ldots, 0.2088)^T$	$(1,1,\ldots 1)^T$	3.2486e-15
$(0.3361, \ldots, 0.2173)^T$	$(1,1,\ldots 1)^T$	4.2236e-15
$(0.1289, \ldots, 0.2394)^T$	$(1,1,\ldots 1)^T$	4.1039e-15

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Table 2 Example 1 numerical results(n=200)

x_0	x^*	$F(x)$
$(0.6351,\dots,0.2566)^T$	$(1,1,\dots 1)^T$	2.7232e-15
$(0.7329,\dots,0.9932)^T$	$(1,1,\dots 1)^T$	2.8321e-15
$(0.4353,\dots,0.2135)^T$	$(1,1,\dots 1)^T$	2.5821e-15
$(0.9921,\dots,0.1563)^T$	$(1,1,\dots 1)^T$	2.5786e-15
$(0.1129,\dots,0.6423)^T$	$(1,1,\dots 1)^T$	1.7659e-15

Example 2 Consider (1)，where $A(\omega)$=

$$\begin{pmatrix} 5+\omega & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 3 \\ 1/2 & 2+\omega & 0 & 1/2 & 1 & 0 & 1 & 0 & 6 & 0 \\ 0 & 1/4 & 7+\omega & 3/4 & 0 & 2 & 0 & 0 & 1/2 & 1/2 \\ 1 & 1 & 2 & 2+\omega & 1/2 & 0 & 3/2 & 2 & 0 & 1 \\ 0 & 0 & 2/5 & 1/4 & 6+\omega & 2 & 0 & 1 & 7/20 & 1 \\ 2 & 1/2 & 4 & 0 & 0 & 1+\omega & 1/2 & 2 & 1 & 0 \\ 0 & 5 & 0 & 2/3 & 0 & 2/3 & 3+\omega & 1/4 & 1 & 5/12 \\ 2 & 1 & 1 & 1 & 1 & 1/2 & 0 & 4+\omega & 1/2 & 0 \\ 1/7 & 5/7 & 0 & 0 & 1 & 0 & 1/7 & 0 & 9+\omega & 0 \\ 3 & 0 & 2 & 1 & 5/2 & 0 & 1/2 & 1/4 & 1/4 & 1+\omega \end{pmatrix}$$

$b(\omega)$=$(10+\omega, 10+\omega, 10+\omega, \cdots, 10+\omega, 10+\omega)^T$,

$\Omega = \{\omega_1, \omega_2\}$=$\{0,1\}$， $p_i = P\{\omega_i \in \Omega\}$=$\frac{1}{2}$， $i = 1,2$.

According to [1], its exact solution is $(1,1,\dots 1)^T$. Algorithm 1 is applied to the above Example 2, taking five sets of initial points within the δ neighborhood of the exact solution x, where x_0 is the initial point and $x^*$is the optimal value. The relevant numerical results are shown in Table 2. We take $\varepsilon = 10^{-6}$， δ=2, λ_k=0.01.

Table 3 Example 2 numerical results

x_0	x^*	$F(x)$
$(0.3381,\dots,0.2346)^T$	$(1.0786,1.1496,\dots 1.0130)^T$	1.2324e-14
$(0.4129,\dots,0.1256)^T$	$(1.0786,1.1496,\dots 1.1130)^T$	2.0123e-14
$(0.2141,\dots,0.5623)^T$	$(1.0786,1.1496,\dots 1.1130)^T$	2.3356e-14
$(0.5236,\dots,0.6422)^T$	$(1.0786,1.1496,\dots 1.1130)^T$	2.5645e-14
$(0.5287,\dots,0.3214)^T$	$(1.0786,1.1496,\dots 1.1130)^T$	1.7649e-14

5.Conclusions

This paper presents a class of stochastic absolute value equations and uses the nonsmooth Newton methods to solve them. Numerical experiments have shown the effectiveness of the algorithm. Due to the wide applications of absolute value equations in engineering, location problems, and other fields, the algorithm design and application research of large-scale stochastic absolute value equations, as well as the general form of absolute value equations will be a problem that needs to be studied.

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