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The property of expected residual minimization method for stochastic tensor complementarity problem

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Abstract: **This paper focus on a deterministic formulation for the stochastic tensor complementarity problem, which called the expected residual minimization formulation. The expected residual minimization formulation aims at minimizing an expected residual defined by nonlinear complementarity functions. The expected residual minimization function based on the min function is proved to be a semi-smooth function. The SC^1 property of the expected residual minimization function based on the min function at strictly feasible points is also proved. The related examples are given to illustrate the property of the ERM function.**

Keywords: Stochastic tensor complementarity problem; Expected residual minimization method; Nonlinear complementarity function

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1. Introduction

In recent years, the tensor complementarity problem and its applications have been widely studied by a wide range of scholars, such as [1-6]. In many practical problems, some elements of tensor complementarity problem may involve a series of uncertain data, such as weather, accident, supply and demand. Due to the influence of these uncertain data，the stochastic tensor complementarity problem (STCP) has been receiving more and more attention in the recent literatures [7-10].

Let $(\Omega, \mathcal{F}, \rho)$ be a probability space, where Ω is a subset of R^n and ρ is a standard probability measure. Here, we assume that ρ is known. Let ω be a stochastic variable in the same sample. In this paper, we mainly focus on the STCP which aims to find a solution $x \in R^n$ such that

$$x \geqslant 0, F(x,\omega) := \mathcal{A}(\omega)x^{m-1} + q(\omega) \geqslant 0,$$
$$x^{\mathrm{T}} F(x,\omega) = 0, \omega \in \Omega, \qquad (1)$$

where $\mathcal{A}(\omega) = (a_{i_1,\ldots,i_m}(\omega))$ denotes an mth-order n-dimensional stochastic real tensor, $q(\omega)$is a stochastic vector in R^n. Clearly, if $m = 2$, the STCP converts to the stochastic linear complementarity problem [11-14]. If Ω only contains a single realization, the STCP reduces to the standard tensor complementarity problem.

Considering the uncertainty of the stochastic variables, it is difficult to find the solution set of STCP directly, just as the stochastic linear complementarity problem. As we all know, the expected residual minimization (ERM) formulation plays an important role in the stochastic linear complementarity problem. The ERM formulation has been widely studied by many scholars in recent years [7-14]. In this paper, the property of the ERM formulation for STCP are mainly considered.

The structure of this paper is as follows: in Section 2, the ERM formulation for STCP is given； In Section 3, we prove that the semi-smooth property of the ERM function for STCP based on the min function; in Section 4, the SC^1 property of the ERM function for STCP based on the min function at strictly feasible points is proved; the relevant conclusions are given in Section 5.

2. Preliminaries

First, we provide a brief summary of the notations used in this paper. $\|\cdot\|$ denotes 2-norm and $E(\cdot)$ stands for the expected function. For vectors $x, y \in R^n$, $\min\{x, y\}$denotes the vector with components $\min\{x_i, y_i\}$, $i \in [n]$. Suppose that m and n be two positive integers with $m, n \geq 2$. $R^{[m,n]}$ denotes all the mth-order n-dimensional stochastic real tensor. For a vector $x := (x_1, \ldots, x_n)^{\mathrm{T}} \in R^n$ and $\mathcal{A}(\omega) \in R^{[m,n]}$ whose elements are $a_{i_1,\ldots,i_m}(\omega)$, $\mathcal{A}(\omega)x^{m-1} \in R^n$ is a column vector, its i-th component is $(\mathcal{A}(\omega)x^{m-1})_i := \sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\ldots i_m(\omega)} x_{i_2} \ldots x_{i_m}$.

The ERM formulation for STCP (1) aims to find a vector $x \in R^n_+$ that minimizes an expected residual function:

$$\min_{x \in R^n_+} G(x) = E\|\Phi(x,\omega)\|^2, \qquad (2)$$

Where $E\|\Phi(x,\omega)\|^2$stands for the expectation of $\|\Phi(x,\omega)\|^2$ and the function $\Phi(x,\omega)$ is defined by:

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$$\Phi(x,\omega) = \begin{pmatrix} \phi(F_1(x,\omega), x_1) \\ \phi(F_2(x,\omega), x_2) \\ \vdots \\ \phi(F_n(x,\omega), x_n)) \end{pmatrix} = \begin{pmatrix} \phi((\mathcal{A}(\omega)x^{m-1} + q(\omega))_1, x_1) \\ \phi((\mathcal{A}(\omega)x^{m-1} + q(\omega))_2, x_2) \\ \vdots \\ \phi((\mathcal{A}(\omega)x^{m-1} + q(\omega))_n, x_n) \end{pmatrix}.$$

Here, $\phi : R^2 \to R$ is an NCP function which satisfies the property

$$\phi(a,b) = 0 \iff a \geq 0,\ b \geq 0,\ ab = 0.$$

Various NCP functions have been studied by scholars for solving ERM problem of complementarity problems [15], such as the Fischer-Burmeister function, the penalized Fischer-Burmeister function and the min function. In the following of this paper, we consider the ERM formulation for STCP based on the min function

$$\phi(a,b) := \min\{a,b\}.$$

3. The semi-smooth property of the ERM function for STCP

In this section, suppose that ω is a continuous stochastic variable with probability density function $\rho(\omega)$, then ERM formulation (2) can be transformed into

$$\min_{x \in R^n_+} G(x) := \int_\Omega \|\Phi(x,\omega)\|^2 \rho(\omega) d\omega,$$

where

$$\Phi(x,\omega) := \min\{x, F(x,\omega)\}.$$

In order to prove $G(x)$ is a semi-smooth function, we first provide a definition which will be used in the following.

Definition 3.1 The function $H : R^n \to R^m$ is said to be semi-smooth at $x \in R^n$ if

$$\lim_{Q \in \partial(x+th'), h' \to h, t \downarrow 0} \{Qh'\}$$

exists for any $h \in R^n$, where $\partial H(x)$ is the generalized Jacobian of function H at x in the sense of Clarke [16]. A function H is said to be a semi-smooth function if it is semi-smooth everywhere on R^n.

We consider an integral function $\gamma : X \to R$, which defined by

$$\gamma(x) := \int_\Omega f(x,\omega)\rho(\omega)d\omega,$$

where $f : X \times \Omega \to R$, X is an open subset of R^n. For $\omega \in \Omega$, let $f_\omega(\cdot) := f(\cdot,\omega)$. Since ρ is a probability measure, ρ is also a finite measure defined on $(\Omega, \mathcal{F})$. Then we give the following proposition.

Proposition 3.1 Let $\bar{x} \in R^n$. Suppose that

(i) there exists a neighborhood $\mathcal{N}(\bar{x})$ of $\bar{x}$ and an integrable function $\kappa : \Omega \to R_+$ such that for a.e. $\omega \in \Omega$ holds

$$|f(x,\omega) - f(\bar{x},\omega)| \leq \kappa(\omega)\|x - \bar{x}\|, \forall x \in \mathcal{N}(\bar{x});$$

(ii) for a.e. $\omega \in \Omega$, $f_\omega(\cdot)$ is semi-smooth at $\bar{x}$;

(iii) there exists a neighborhood $\mathcal{N}(0)$ of $0 \in R^n$ and an integrable function $\gamma : \Omega \to R_+$such that for a.e. $\omega \in \Omega$ holds

$$\frac{|f'_\omega(\bar{x}+h;h) - f'_\omega(\bar{x};h)|}{\|h\|} \leq \gamma(\omega), \forall h \in \mathcal{N}(0)\ ,$$

where $f'_\omega(\bar{x};h)$ denotes the directional derivative of $f(\bar{x},\omega)$ along direction h.

Then $\gamma(\cdot)$ is semi-smooth at $\bar{x}$.

Assumption 3.1 Let $\bar{x} \in R^n$.

(i) There exists a neighborhood $\mathcal{N}(\bar{x})$ of $\bar{x}$ and an integrable function $g(\omega)$ such that

$$\sum_{i=1}^n \{|F_i(x,\omega)|^2 + \|\nabla_x F_i(x,\omega)\|^2\} \leq g(\omega),\ a.e.\ \omega \in \Omega$$

for all $x \in \mathcal{N}(\bar{x})$;

(ii) There exists a neighborhood $\mathcal{N}(\bar{x})$ of $\bar{x}$ and a function $\beta(\omega)$ satisfying

$$\int_\Omega (\beta(\omega))^2 \rho(\omega) d\omega < +\infty,$$

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such that

$$\|\nabla_x F_i(x,\omega) - \nabla_x F_i(x',\omega)\| \le \beta(\omega)\|x - x'\|,$$

for every $i = 1, 2, \ldots, n$ and any $x, x' \in \mathcal{N}(\bar{x})$.

Next, we will study the semi-smooth property of the ERM function.

Theorem 3.1 Let $\bar{x} \in R^n$.Suppose that function $F(x,\omega)$ holds for the Assumption 3.1 (i), then $G(x)$ is a semi-smooth function on R^n.

Proof Consider a neighborhood of $\bar{x}$,

$$\mathcal{N}_r(\bar{x}) = \{x | \|x - \bar{x}\| \le r\}(r > 0).$$

For $\forall x \in \mathcal{N}_r(\bar{x})$, by integral mean value theorem and Assumption 3.1 (i), we have

$$\begin{aligned}\|\Phi_{\min}(x,\omega)\| &= \|\min\{x, F(x,\omega)\}\| \\ &\le \|x\| + \|F(x,\omega)\| \\ &\le \|\bar{x}\| + r + \|F(\bar{x},\omega)\| + r\|\nabla F(\hat{x},\omega)\| \\ &\le \|\bar{x}\| + r + \sqrt{g(\omega)} + r\sqrt{g(\omega)} \\ &\le (1 + \sqrt{g(\omega)})(1 + \|\bar{x}\| + r),\end{aligned}$$

where $\hat{x}$ (it is dependent on ω) is in the open segment connecting x and $\bar{x}$. Denote

$$\Theta(x,\omega) = \frac{1}{2}\|\Phi_{\min}(x,\omega)\|^2,$$

$$\begin{aligned}&|\Theta(x,\omega) - \Theta(\bar{x},\omega)| \\ &= \frac{1}{2}|\|\Phi_{\min}(x,\omega)\|^2 - \|\Phi_{\min}(\bar{x},\omega)\|^2| \\ &\le \frac{1}{2}\|\Phi_{\min}(x,\omega) + \Phi_{\min}(\bar{x},\omega)\|\|\Phi_{\min}(x,\omega) - \Phi_{\min}(\bar{x},\omega) \\ &\le (1 + \sqrt{g(\omega)})(1 + \|\bar{x}\| + r)\|\Phi_{\min}(x,\omega) - \Phi_{\min}(\bar{x},\omega)\|,\end{aligned}$$

then,

and

$$\begin{aligned}&\|\Phi_{\min}(x,\omega) - \Phi_{\min}(\bar{x},\omega)\| \\ &= \|\min\{x, F(x,\omega)\} - \min\{\bar{x}, F(\bar{x},\omega)\}\| \\ &\le \|x - \bar{x}\| + \|F(x,\omega) - F(\bar{x},\omega)\| \\ &= \|x - \bar{x}\| + \|\nabla F(\widetilde{x},\omega)\|\|x - \bar{x}\| \\ &\le (1 + \sqrt{g(\omega)})\|x - \bar{x}\|,\end{aligned}$$

where $\widetilde{x}$ (it is dependent on ω) is in the open segment connecting x and $\bar{x}$. Then,

$|\Theta(x,\omega) - \Theta(\bar{x},\omega)| \le (1 + \sqrt{g(\omega)})^2(1 + \|\bar{x}\| + r)\|x - \bar{x}\|$.

Let $\kappa(\omega) = (1 + \sqrt{g(\omega)})^2(1 + \|\bar{x}\| + r)$, we get

$$|\Theta(x,\omega) - \Theta(\bar{x},\omega)| \le \kappa(\omega)\|x - \bar{x}\|,$$

this means Proposition 3.1 (i) holds for $\Theta(x,\omega)$.

For $\omega \in \Omega$, $\min\{x_i, F_i(x,\omega)\}$ is a piecewise smooth function, so it is a semi-smooth function. Because of the composite of semi-smooth functions is also a semi-smooth function, $\Theta(x,\omega)$ is a semi-smooth function, i.e., $\Theta(x,\omega)$ satisfies Proposition 3.1 (ii).

In the following, taking a neighborhood $\mathcal{N}(0) = \{h | \|h\| \le \delta\}$ of 0 such that

$$\begin{aligned}&\Theta'_\omega(\bar{x} + h; h) \\ &= \sum_{i \in \alpha_\omega(x)} F_i(\bar{x} + h,\omega)(\nabla F_i(\bar{x} + h,\omega)^{\mathsf{T}} h)_i + \sum_{i \in \gamma_\omega(x)} (\bar{x} + h)_i h_i \\ &+ \sum_{i \in \beta_\omega(x)} (\bar{x} + h)_i \min\{h_i, (\nabla F_i(\bar{x} + h,\omega)^{\mathsf{T}} h)_i\},\end{aligned}$$

where

$$\begin{aligned}\alpha_\omega(x) &= \{i : x_i \ge F_i(x,\omega)\}, \\ \beta_\omega(x) &= \{i : x_i = F_i(x,\omega)\}, \\ \gamma_\omega(x) &= \{i : x_i \le F_i(x,\omega)\}.\end{aligned}$$

By Assumption 3.1 (i), we obtain

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$$
\begin{aligned}
&\|\Theta_{\omega}^{'}(\bar{x}+h;h)\| \\
\leq&\|\sum_{i\in\alpha_{\omega}(x)} F_i(\bar{x}+h,\omega)(\nabla F_i(\bar{x}+h,\omega)^{\mathsf{T}}h)_i\| \\
&+\|\sum_{i\in\gamma_{\omega}(x)} (\bar{x}+h)_i h_i\| \\
&+\|\sum_{i\in\beta_{\omega}(x)} (\bar{x}+h)_i \min\{h_i,(\nabla F_i(\bar{x}+h,\omega)^{\mathsf{T}}h)_i\}\| \\
\leq&\|F_i(\bar{x}+h,\omega)\|\|(\nabla F_i(\bar{x}+h,\omega)\|\|h\|+\|\bar{x}+h\|\|h\| \\
&+\|\bar{x}+h\|(\|h\|+\|(\nabla F_i(\bar{x}+h,\omega)\|\|h\|) \\
\leq&g(\omega)\|h\|+(\|\bar{x}\|+\delta)\|h\|+(\|\bar{x}\|+\delta)(1+\sqrt{g(\omega)})\|h\| \\
=&[g(\omega)+(\|\bar{x}\|+\delta)+(\|\bar{x}\|+\delta)(1+\sqrt{g(\omega)})]\|h\| \\
\leq&(g(\omega)+2+\sqrt{g(\omega)})(1+\|\bar{x}\|+\delta)\|h\|.
\end{aligned}
$$

Let $\gamma(\omega)=2(g(\omega)+2+\sqrt{g(\omega)})(1+\|\bar{x}\|+\delta)$, we have

$$
\begin{aligned}
\frac{|\Theta_{\omega}^{'}(\bar{x}+h;h)-\Theta_{\omega}^{'}(\bar{x};h)|}{\|h\|} &\leq \frac{\|\Theta_{\omega}^{'}(\bar{x}+h;h)\|+\|\Theta_{\omega}^{'}(\bar{x};h)\|}{\|h\|} \\
&\leq \gamma(\omega),
\end{aligned}
$$

for all $h\in\mathcal{N}(0)$.Therefore, the above formula satisfies Proposition 3.1 (iii).

By Proposition 3.1, we can conclude that function $G(x)$ is semi-smooth at $\bar{x}$, for all $\bar{x}\in R^n$. Thus, $G(x)$ is a semi-smooth function on R^n.

The proof is complete.

Example 3.1 Let stochastic variable ω is uniformly distributed on the sample space $\Omega=[0,1]$. Consider STCP (1) with $\mathcal{A}(\omega)\in R^{[3,2]}$, where $a_{111}=\omega$, $a_{222}=\omega$ and all other $a_{i_1i_2i_3}=0$, $q(\omega)=(\omega,-\omega)^{\mathsf{T}}$.

$$
F(x,\omega)=\mathcal{A}(\omega)x^{m-1}+q(\omega)=\begin{pmatrix}\omega x_1^2+\omega\\ \omega x_2^2-\omega\end{pmatrix},
$$

$$
\nabla F_1(x,\omega)=\begin{pmatrix}2\omega x_1\\ 0\end{pmatrix},
$$

$$
\nabla F_2(x,\omega)=\begin{pmatrix}0\\ 2\omega x_2\end{pmatrix}.
$$

It is easy to verify Assumption 3.1 (i) holds. By calculation, we obtain that

$$
\begin{aligned}
G(x)&=E\|\Phi(x,\omega)\|^2\\
&=\int_{\Omega}\|\Phi(x,\omega)\|^2\rho(\omega)d\omega\\
&=\int_0^1[(\frac{x_1+F_1(x,\omega)-|x_1-F_1(x,\omega)|}{2})^2\\
&+(\frac{x_2+F_2(x,\omega)-|x_2-F_2(x,\omega)|}{2})^2]d\omega.
\end{aligned}
$$

Next, the differentiability of $G(x)$ are discussed in two cases.

Case 1: $x_1=F_1(x,\omega)$ or $x_2=F_2(x,\omega)$, $G(x)$ is non-differentiable.

Case 2:

(1) For $x_1>F_1(x,\omega)$ and $x_2>F_2(x,\omega)$,

$$
G(x)=x_1^2+x_2^2,\ \nabla G(x)=\begin{pmatrix}2x_1\\ 2x_2\end{pmatrix}.
$$

(2) For $x_1>F_1(x,\omega)$ and $x_2<F_2(x,\omega)$,

$$
G(x)=x_1^2+\frac{1}{3}(x_2^2-1)^2,\ \nabla G(x)=\begin{pmatrix}2x_1\\ \frac{4}{3}x_2(x_2^2-1)\end{pmatrix}.
$$

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(3) For $x_1 < F_1(x,\omega)$ and $x_2 > F_2(x,\omega)$,

$$G(x) = \frac{1}{3}(x_1^2+1)^2 + x_2^2,\ \nabla G(x) = \begin{pmatrix} \frac{4}{3}x_1(x_1^2+1) \\ 2x_2 \end{pmatrix}.$$

(4) For $x_1 < F_1(x,\omega)$ and $x_2 < F_2(x,\omega)$,

$$G(x) = \frac{1}{3}(x_1^2+1)^2 + \frac{1}{3}(x_2^2-1)^2,\ \nabla G(x) = \begin{pmatrix} \frac{4}{3}x_1(x_1^2+1) \\ \frac{4}{3}x_2(x_2^2-1) \end{pmatrix}.$$

Thus, $G(x)$ is a semi-smooth function on R^n.

4. The SC^1 property of the ERM formulation for STCP

Definition 4.1 A function $f : R^n \to R$ is said to be an SC^1 function if f is continuously differentiable and its gradient is semi-smooth.

The SC^1property is a crucial condition for superlinear convergence of the SQP method for solving nonlinear programming problems that are once, but not twice, continuously differentiable [17]. From Theorem 3.1, we know that $G(x)$ is a semi-smooth function on R^n but not continuously differentiable. Hence, we will consider the SC^1 property of $G(x)$ at the strictly feasible point.

Definition 4.2 For a given $\omega \in \Omega$, if $x > 0$ and $F(x,\omega) > 0$, then x is called the strictly feasible point of STCP (1).

Theorem 4.1 Suppose that $\bar{x}$ is the strictly feasible point of STCP (1), if Assumption 3.1 holds, then function $G(x)$ is an SC^1 function at $\bar{x}$.

Proof In the following proof, we use F_i and ∇F_i denote $F_i(x,\omega)$ and $\nabla F_i(x,\omega)$ respectively, and define

$$G(x) = \sum_{i=1}^{n} g_i(x),$$

where

$$g_i(x) = \int_\Omega \psi_i(x,\omega)\rho(\omega)d\omega, i = 1,2,\ldots,n,$$

and

$$\begin{aligned}\psi_i(x,\omega) &= (\min\{x_i, F_i\})^2 \\ &= (\frac{x_i + F_i - \sqrt{(x_i - F_i)^2}}{2})^2 \\ &= \frac{x_i^2 + F_i^2 - (x_i + F_i)|x_i - F_i|}{2}.\end{aligned}$$

By simple computation, we have

$$\nabla g_i(x) = \int_\Omega \nabla_x \psi_i(x,\omega)\rho(\omega)d\omega,$$

where

$$\nabla_x \psi_i(x,\omega) = x_i e_i + F_i \nabla F_i - \frac{1}{2}\nabla_x q_i(x,\omega),$$

and

$$q_i(x,\omega) = (x_i + F_i)|x_i - F_i|.$$

Let $\Omega_1 = \{\omega | x_i \geq F_i(x,\omega)\}$,$\Omega_2 = \{\omega | x_i \leq F_i(x,\omega)\}$,
then, $\Omega = \Omega_1 \cup \Omega_2$.
Therefore,

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$$\begin{aligned}
&\nabla g_i(x)\\
&= \int_{\Omega} \nabla_x \psi_i(x,\omega)\rho(\omega)d\omega\\
&= \int_{\Omega_1} \nabla_x \psi_i(x,\omega)\rho(\omega)d\omega + \int_{\Omega_2} \nabla_x \psi_i(x,\omega)\rho(\omega)d\omega\\
&= \int_{\Omega_1} 2F_i\nabla F_i\rho(\omega)d\omega + \int_{\Omega_2} 2x_i e_i\rho(\omega)d\omega.
\end{aligned}$$

Let $\mu(x,\omega) = 2F_i\nabla F_i$, $\nu(x) = 2x_i e_i$, then

$$\nabla g_i(x) = \int_{\Omega_1} \mu(x,\omega)\rho(\omega)d\omega + \int_{\Omega_2} \nu(x)\rho(\omega)d\omega.$$

According to Assumption 3.1 (i), we know that

$$|\mu(x,\omega)| \le 2\|F_i\|\|\nabla F_i\| \le 2g(\omega), \forall x \in \mathcal{N}(\bar{x}),$$

for all $\omega \in \Omega$, where $\bar{x}$ is the strictly feasible point. By the Lebesgue dominated convergence theorem,

$$\begin{aligned}
&\lim_{x\to\bar{x}} \nabla g_i(x)\\
&= \lim_{x\to\bar{x}} \int_{\Omega_1} \mu(x,\omega)\rho(\omega)d(\omega) + \lim_{x\to\bar{x}} \int_{\Omega_2} \nu(x)\rho(\omega)d(\omega)\\
&= \int_{\Omega_1} \lim_{x\to\bar{x}} \mu(x,\omega)\rho(\omega)d(\omega) + \int_{\Omega_2} \lim_{x\to\bar{x}} \nu(x)\rho(\omega)d(\omega)\\
&= \int_{\Omega_1} \mu(\bar{x},\omega)\rho(\omega)d(\omega) + \int_{\Omega_2} \nu(\bar{x})\rho(\omega)d(\omega)\\
&= \nabla g_i(\bar{x}),
\end{aligned}$$

thus, $\nabla g_i(x)$ is continuous at $\bar{x}$, the function $G(x)$ is continuously differentiable at $\bar{x}$.

In order to prove that $G(x)$ is an SC^1 function at the strictly feasible point $\bar{x}$, we only need to show that $\nabla g_i(x)$ is a semi-smooth function at $\bar{x}$. It is obvious that $\int_{\Omega_2} \nu(x)\rho(\omega)d\omega$ is a semi-smooth function at $\bar{x}$. Next, we will prove that $\int_{\Omega_1} \mu(x,\omega)\rho(\omega)d\omega$ is a semi-smooth function at $\bar{x}$.

From Assumption 3.1 and the integral mean value theorem, we can get

$$\begin{aligned}
&|\mu(x,\omega) - \mu(\bar{x},\omega)|\\
&= |2F_i\nabla F_i - 2\bar{F}_i\nabla\bar{F}_i|\\
&= 2|F_i\nabla F_i - F_i\nabla\bar{F}_i + F_i\nabla\bar{F}_i - \bar{F}_i\nabla\bar{F}_i|\\
&= 2|F_i(\nabla F_i - \nabla\bar{F}_i) + \nabla\bar{F}_i\frac{F_i - \bar{F}_i}{x_i - \bar{x}_i}x_i - \bar{x}_i|\\
&\le 2\|F_i\|\|\nabla F_i - \nabla\bar{F}_i\| + 2\|\nabla\bar{F}_i\|\|\nabla\widehat{F}_i\|\|x_i - \bar{x}_i\|\\
&\le 2\sqrt{g(\omega)}\beta(\omega)\|x - \bar{x}\| + 2g(\omega)\|x - \bar{x}\|\\
&= 2\sqrt{g(\omega)}(\beta(\omega) + \sqrt{g(\omega)})\|x - \bar{x}\|,\ \forall x \in \mathcal{N}(\bar{x}),\ \forall\omega \in \Omega
\end{aligned}$$

where $\widehat{x}$ (it is dependent on ω) is in the open segment connecting x and $\bar{x}$. Let

$$\kappa(\omega) = 2\sqrt{g(\omega)}(\beta(\omega) + \sqrt{g(\omega)}),$$

then Proposition 3.1 (i) holds .

Because of the function F_i is a smooth function and ∇F_i is a semi-smooth function, we know that $\mu(x,\omega)$ is a semi-smooth function at $\bar{x}$, i.e., Proposition 3.1 (ii) holds.

At the strictly feasible point $x \in \mathcal{N}(\bar{x})$, define

$$\mu_j(x,\omega) = 2F_i(\nabla F_i)_j$$

where $(\nabla F_i)_j$denotes the j-th item of ∇F_i. Denote $\mu'_{j,\omega}(x;h)$ is the directional derivative of $\mu_j(x,\omega)$ along any direction $h \in \mathcal{N}(0)$.

By simple computation, we have

$$\mu'_{j,\omega}(x;h) = 2(\nabla F_i(\nabla F_i)_j + F_i\nabla(\nabla F_i)_j)h.$$

According to Assumption 3.1, then

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$$\begin{aligned}
&\frac{|\mu'_{j,\omega}(x+h;h)-\mu'_{j,\omega}(x;h)|}{\|h\|}\\
&\le \frac{|\mu'_{j,\omega}(x+h;h)|+|\mu'_{j,\omega}(x;h)|}{\|h\|}\\
&\le 4\|\nabla F_i(\nabla F_i)_j + F_i\nabla(\nabla F_i)_j\|\\
&\le 4\|\nabla F_i\|\|(\nabla F_i)_j\| + 4\|F_i\|\|\nabla(\nabla F_i)_j\|\\
&\le 4\sqrt{g(\omega)}\sqrt{g(\omega)} + 4\sqrt{g(\omega)}\beta(\omega)\\
&= 4g(\omega) + 4\sqrt{g(\omega)}\beta(\omega).
\end{aligned}$$

Let $\gamma(\omega) = 4g(\omega) + 4\sqrt{g(\omega)}\beta(\omega)$. Hence, Proposition 3.1 (iii) holds for the function $\mu(x,\omega)$.

By Proposition 3.1, we can see that the function $\int_{\Omega_1}\mu(x,\omega)\rho(\omega)d\omega$ is a semi-smooth function at $\bar{x}$. Thus, $\nabla g_i(x)(i=1,...n)$ is a semi-smooth function at $\bar{x}$.

In summary, $G(x)$ is continuously differentiable at $\bar{x}$ and its gradient is semi-smooth at $\bar{x}$. By Definition 4.1, we obtain that $G(x)$ is an SC^1 function at the strictly feasible point $\bar{x}$.

The proof is complete.

Theorem 4.1 shows that the ERM function for STCP based on the min function is an SC^1 function at all strictly feasible points.

Example 4.1 Let stochastic variable ω is uniformly distributed on the sample space $\Omega = [0,1]$. Consider STCP (1) with $\mathcal{A}(\omega) \in R^{[4,2]}$, where $a_{1111} = \omega$, $a_{2222} = \omega$ and all other $a_{i_1i_2i_3i_4} = 0$, $q(\omega) = (-\omega,-\omega)^{\mathrm{T}}$.

$$F(x,\omega) = \mathcal{A}(\omega)x^{m-1} + q(\omega) = \begin{pmatrix}\omega x_1^3 - \omega\\ \omega x_2^3 - \omega\end{pmatrix},$$

$$\nabla F_1(x,\omega) = \begin{pmatrix}3\omega x_1^2\\ 0\end{pmatrix},$$

$$\nabla F_2(x,\omega) = \begin{pmatrix}0\\ 3\omega x_2^2\end{pmatrix}.$$

After verification, Assumption 3.1 holds. Then

$$\begin{aligned}
G(x) &= E\|\Phi(x,\omega)\|^2\\
&= \int_\Omega \|\Phi(x,\omega)\|^2\rho(\omega)d\omega\\
&= \int_0^1 [(\frac{x_1 + F_1(x,\omega) - |x_1 - F_1(x,\omega)|}{2})^2\\
&+ (\frac{x_2 + F_2(x,\omega) - |x_2 - F_2(x,\omega)|}{2})^2]d\omega.
\end{aligned}$$

From the above formula, we know that $G(x)$ is continuously differentiable at strictly feasible points $\bar{x}$. The following situations are hold respectively.

(1) For $\bar{x}_1 \ge F_1(\bar{x},\omega)$ and $\bar{x}_2 \ge F_2(\bar{x},\omega)$,

$$G(\bar{x}) = \bar{x}_1^2 + \bar{x}_2^2,\ \nabla G(\bar{x}) = \begin{pmatrix}2\bar{x}_1\\ 2\bar{x}_2\end{pmatrix}.$$

(2) For $\bar{x}_1 \ge F_1(\bar{x},\omega)$ and $\bar{x}_2 < F_2(\bar{x},\omega)$,

$$G(\bar{x}) = \bar{x}_1^2 + \frac{1}{3}(\bar{x}_2^3 - 1)^2,\ \nabla G(\bar{x}) = \begin{pmatrix}2\bar{x}_1\\ \\ 2\bar{x}_2^2(\bar{x}_2^3 - 1)\end{pmatrix}.$$

(3) For $\bar{x}_1 < F_1(\bar{x},\omega)$ and $\bar{x}_2 \ge F_2(\bar{x},\omega)$,

$$G(\bar{x}) = \frac{1}{3}(\bar{x}_1^3 - 1)^2 + \bar{x}_2^2,\ \nabla G(\bar{x}) = \begin{pmatrix}2\bar{x}_1^2(\bar{x}_1^3 - 1)\\ \\ 2\bar{x}_2\end{pmatrix}.$$

(4) For $\bar{x}_1 < F_1(\bar{x},\omega)$ and $\bar{x}_2 < F_2(\bar{x},\omega)$,

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$$G(\bar{x}) = \frac{1}{3}(\bar{x}_1^3 - 1)^2 + \frac{1}{3}(\bar{x}_2^3 - 1)^2,\ \nabla G(\bar{x}) = \begin{pmatrix} 2\bar{x}_1^2(\bar{x}_1^3 - 1) \\ 2\bar{x}_2^2(\bar{x}_2^3 - 1) \end{pmatrix}.$$

Therefore, $G(x)$ is an SC^1 function at the strictly feasible point $\bar{x}$.

5. Conclusions

This paper consider the STCP and transform the STCP into the ERM formulation which is defined by the min function. Some properties of the ERM function for the STCP has been proved, such as the semi-smooth property of the ERM function based on the min function and the SC^1 property of the ERM function based on the min function at the strictly feasible points.

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