Journal of Computational Science & Engineering
Available at http://www.asocse.org ISSN 1710-4068 57(2022) 32-37

Design of H_∞ Controller for Single Inverted Pendulum Based on Output Feedback

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Abstract: For a single inverted pendulum control system, the design method of a class of output feedback H_∞ controller and the performance of the robust control are studied, then the H_∞ controller design is studied for the control system mentioned above based on LMI optimization, and a sufficient condition is derived in terms of linear matrix inequalities. The designed controller guarantees the stability of the single inverted pendulum control system, and has good anti-interference ability. Numerical example is given to illustrate the effectiveness of the proposed method.

Keywords: Single inverted pendulum; H_∞ control; Linear matrix inequality

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Received 03 August 2022, Revised 15 August 2022, Accepted 22 August 2022

1.Introduction

The inverted pendulum control system is a kind of ideal control theory research and experimental equipment with high order, instability, multivariable, non-linearity and strong coupling. It is also an effective platform for researchers in control field to verify new theories and methods [1-2]. At present, the related research results on inverted pendulum control system have been widely used in various high-tech fields, such as aerospace, robot control and so on. Therefore, the research on inverted pendulum control system has both theoretical value and engineering significance [3].

Now, the research results of inverted pendulum control system are very rich. The PID and LQR control methods for single inverted pendulum systems were studied in [4, 5]. The observer-based method has been used to study the inverted pendulum system in [6]. The H_∞ control problem of single inverted pendulum system based on state feedback is studied by using Riccati equation and LMI optimization method respectively in [7, 8]. It has been considered that the control method based on state feedback has some limitations because only the car displacement and the swing rod angle can be directly measured in the inverted pendulum system variables, while the car speed and the swing rod speed can not be directly measured. In [9, 10], the problem of output feedback design is studied for single inverted pendulum and double inverted pendulum respectively. However, the adverse effects of external uncertainty disturbance on robust control performance are not fully considered.

In this paper, the frictional resistance between the car and the guide rail of the inverted pendulum is regarded as external interference for the single inverted pendulum control system. According to the performance requirements of the H_∞ control, the design of a class of controller based on output feedback is studied by using linear matrix inequality (LMI) method. The sufficient conditions for the existence of the controller are given by using LMI optimization technology, and the parameters of the controller to be designed are solved by using MATLAB toolbox.

2.Problem formulation

The linear model of single inverted pendulum [11]:

$$\begin{cases} \dot{x}(t) = Ax(t) + B_1\omega(t) + B_2u(t) \\ y(t) = C_1x(t) + D_{11}\omega(t) + D_{12}u(t) \\ \psi(t) = C_2x(t) + D_{21}\omega(t) + D_{22}u(t) \end{cases} \quad (1)$$

where $x(t) \in R^n$ is the state vector; $y(t) \in R^p$ is the controlled output signal; $\psi(t) \in R^p$ is the signal to be estimated; $\psi(t) \in R^p$ is the controlled input signal; $\omega(t) \in R^q$ is the external interference signal; A, B_1, B_2, C_1, D_{11}

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, D_{12} , C_2 , D_{21} and D_{22} are known real constant matrices with appropriate dimensions.

Since the car speed and pendulum rod swing speed of single inverted pendulum system can not be observed directly, the H_∞ controller $u(t)=K(s)\psi(t)$ based on output feedback will be designed. The state space implementation of the controller to be designed is as follows:

$$\begin{cases}\dot{x}_c(t)=A_c x_c(t)+B_c\psi(t)\\ u(t)=C_c x_c(t)+D_c\psi(t)\end{cases} \quad (2)$$

where $x_c(t)\in R^n$ is the state variable of controller; A_c , B_c , C_c and D_c are the gain matrices of controller to be designed.

Remark 1. In order to simplify the calculation, reduce the difficulty of derivation and without loss of generality, the matrix $D_{22}=0$ in the following design algorithm.

Define the augmented state vector $x_\xi(t)=\left[x^T(t) \quad x_c^T(t)\right]^T$,then the closed-loop control system of single inverted pendulum can be described as

$$\begin{cases}\dot{x}_\xi(t)=A_\xi x_\xi(t)+B_\xi\omega(t)\\ y(t)=C_\xi x_\xi(t)+D_\xi\omega(t)\end{cases} \quad (3)$$

where $A_\xi=\begin{bmatrix}A+B_2D_cC_2 & B_2C_c\\ B_cC_2 & A_c\end{bmatrix}$, $B_\xi=\begin{bmatrix}B_1+B_2D_cD_{21}\\ B_cD_{21}\end{bmatrix}$, $C_\xi=\begin{bmatrix}C_1+D_{12}D_cC_2 & D_{12}C_c\end{bmatrix}$, $D_\xi=D_{11}+D_{12}D_cD_{21}$.

The design goal of this paper is to design a controller of the form (2) so that the closed-loop control system (3) satisfies:

(1)when $\omega(t)=0$, then the system is asymptotic stable;

(2)Under zero initial conditions, for any non-zero $\omega(t)\in L_2[0, \infty)$, the system has H_∞ control performance, that is, the closed-loop transfer function satisfies $\|T_{\omega y}(s)\|_\infty<\gamma$, $\gamma>0$.

3. Design of H_∞ controller

Conclusion 1.For a given $\gamma>0$, the closed-loop control system (3) is asymptotic stable and has a given H_∞ performance, if there exist positive definite symmetric matrix P such that equation (4) holds.

$$\begin{bmatrix}A_\xi^TP+PA_\xi & PB_\xi & C_\xi^T\\ B_\xi^TP & -\gamma^2I & D_\xi^T\\ C_\xi & D_\xi & -I\end{bmatrix}<0 \quad (4)$$

Proof: Select Lyapunov function of the closed-loop control system (3) as follows:

$V(x(t),t)=x_\xi^T(t)Px_\xi(t)$ (5)

(i)The proof of the asymptotic stability of the closed-loop control system (3) with $\omega(k)$=0 .

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$$\begin{aligned}\dot{V}\left(x(t),t\right)&=\dot{x}_{\xi}^{T}(t)Px_{\xi}(t)+x_{\xi}^{T}(t)P\dot{x}_{\xi}(t)\\&=x_{\xi}^{T}(t)\left(A_{\xi}^{T}P+PA_{\xi}\right)x_{\xi}(t)\end{aligned}\tag{6}$$

By using Schur complement lemma, it is easy to get $A_{\xi}^{T}P+PA_{\xi}<0$ from equation (4). Therefore, for any $x_{\xi}(t)\neq 0$, there is $\dot{V}\left(x(t),t\right)<0$. According to the Lyapunov stability theorem, the closed-loop control system is asymptotically stable.

(ii)The proof of the H_{∞} performance of the closed-loop control system (3) for all nonzero $\omega(t)\neq 0$.

$$\begin{aligned}J_{T}&=\int_{0}^{T}\left[y^{T}(t)y(t)-\gamma^{2}\omega^{T}(t)\omega(t)\right]dt\\&\leq\int_{0}^{T}\left[y^{T}(t)y(t)-\gamma^{2}\omega^{T}(t)\omega(t)+\dot{V}\left(x(t),t\right)\right]dt\\&=\int_{0}^{T}\left(\begin{bmatrix}x_{\xi}(t)\\ \omega(t)\end{bmatrix}^{T}\Lambda\begin{bmatrix}x_{\xi}(t)\\ \omega(t)\end{bmatrix}\right)dt\end{aligned}\tag{7}$$

where $\Lambda=\begin{bmatrix}C_{\xi}^{T}\\ D_{\xi}^{T}\end{bmatrix}\begin{bmatrix}C_{\xi} & D_{\xi}\end{bmatrix}+\begin{bmatrix}A_{\xi}^{T}X+XA_{\xi} & XB_{\xi}\\ B_{\xi}^{T}X & -\gamma^{2}I\end{bmatrix}$.

By using Schur complement lemma, we can get that equation (4) is equivalent to $\Lambda<0$, so that $\left\|T_{\omega y}(s)\right\|_{\infty}<\gamma$ holds, that is, the closed-loop control system has a given H_{∞} performance.

This ends the proof.

On the basis of conclusion 1, the design method of the H_{∞} controller will be further given.

Conclusion 2.For a given $\gamma>0$, the closed-loop control system (3) is asymptotic stable and has a given H_{∞} performance, if there exist positive definite symmetric matrices X and Y ,matrices $\hat{A}$, $\hat{B}$, $\hat{C}$ and $\hat{D}$ such that equation (8) holds.

$$\begin{bmatrix}AX+XA^{T}+B_{2}\hat{C}+C^{T}B_{2}^{T} & * & * & *\\ A^{T}+\hat{A}+C_{2}^{T}\hat{D}^{T}B_{2}^{T} & YA+A^{T}Y+BC_{2}+C_{2}^{T}B^{T} & * & *\\ B_{1}^{T}+D_{21}^{T}\hat{D}^{T}B_{2}^{T} & B_{1}^{T}Y+D_{21}^{T}B^{T} & -\gamma^{2}I & *\\ C_{1}X+D_{12}\hat{C} & C_{1}+D_{12}\hat{D}C_{2} & D_{11}+D_{12}DD_{21} & -I\end{bmatrix}<0\tag{8}$$

Proof: Partitioning matrices P and P^{-1} into the following block forms:

$$P=\begin{bmatrix}Y & U\\ U^{T} & \vartheta_{1}\end{bmatrix},\ P^{-1}=\begin{bmatrix}X & V\\ V^{T} & \vartheta_{2}\end{bmatrix}$$

where $X\in R^{n\times n}$ and $Y\in R^{n\times n}$ are symmetric positive definite matrices, U and V are full rank matrices, ϑ_{1} and ϑ_{2} are arbitrary matrices and are determined by $P*P^{-1}=I$.

Define the invertible matrices as follows:

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$$\eta_1 = \begin{bmatrix} X & I \\ V^T & 0 \end{bmatrix}, \eta_2 = \begin{bmatrix} I & Y \\ 0 & U^T \end{bmatrix}$$

which implies that $P\eta_1 = \eta_2$.

Performing congruence transformation to equation (4) by the transformational matrix Ξ_1 and defining the following matrix variables:

$$\begin{aligned} &\hat{A} = Y\left(AX + B_2 D_c C_2 X\right) + UB_c C_2 X + YB_2 C_c V^T + UA_c V^T \\ &\hat{B} = YB_2 D_c + UB_c, \quad \hat{C} = D_c C_2 X + C_c V^T \quad D = D_c \end{aligned} \tag{9}$$

then equation (9) is equivalent to equation (8), where $\Xi_1 = diag\{\eta_1, \quad I, \quad I\}$.

$$\begin{bmatrix} \eta_1^T A_\xi^T \eta_2 + \eta_2^T A_\xi \eta_1 & \eta_2^T B_\xi & \eta_1^T C_\xi^T \\ B_\xi^T \eta_2 & -\gamma^2 I & D_\xi^T \\ C_\xi \eta_1 & D_\xi & -I \end{bmatrix} < 0 \tag{10}$$

This ends the proof.

The controller matrix parameters can be obtained from equation (9):

$$\begin{aligned} &A_c = U^{-1}\psi_1\left(V^T\right)^{-1} \\ &B_c = U^{-1}\psi_2 \\ &C_c = \psi_3\left(V^T\right)^{-1} \\ &D_c = \hat{D} \end{aligned} \tag{11}$$

where $\psi_1 = \hat{A} - YAX - \hat{B}C_2X - YB_2C + YB_2DC_2X$, $\psi_2 = \hat{B} - YB_2D$, $\psi_3 = \hat{C} - \hat{D}C_2X$

In order to solve the problem that the matrix needed to construct the controller parameter matrix is not included in equation (8), the transfer function $T_{u\psi}(s)$ of controller (2) is used.

$$T_{u\psi}(s) = C_c\left(sI - A_c\right)^{-1} B_c + D_c \tag{12}$$

Substituting equation (11) into equation (12) and simplifying it:

$$T_{u\psi}(s) = \psi_3\left(zUV^T - \psi_1\right)^{-1}\psi_2 + \hat{D} \tag{13}$$

According to $PP^{-1} = I$, it is easy to deduce $UV^T = I - YX$, then substitute it into equation (13) and simplify it:

$$\begin{aligned} T_{u\psi}(s) &= \psi_3\left(s\left(I - YX\right) - \psi_1\right)^{-1}\psi_2 + \hat{D} \\ &= \psi_3\left(sI - \left(I - YX\right)^{-1}\psi_1\right)^{-1}\left(I - YX\right)^{-1}\psi_2 + \hat{D} \end{aligned} \tag{14}$$

The matrix parameters of the controller can be obtained from the sum of equation (12) and equation (14) as shown in equation (15).

$$A_c = \left(I - YX\right)^{-1}\psi_1, B_c = \left(I - YX\right)^{-1}\psi_2, C_c = \psi_3, D_c = \hat{D} \tag{15}$$

Remark 2: On the basis of conclusion 2, the design of H_∞ controller can be further optimized. The sub-optimal H_∞ controller can be obtained by solving the following optimization problems with $g = \gamma^2$ as shown in equation (16).

$$\min_{\substack{X,Y,\hat{A},\hat{B},C,D \\ s.t.(8)}} g \tag{16}$$

The sub-optimal H_∞ attenuation levels are given by $\gamma_{opt} = \sqrt{g^*}$, where g^* are the sub-optimal solutions of the

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corresponding convex optimization problems.

Through the above transformation, the design problem of the H_∞ controller based on output feedback is transformed into a convex optimization problem with LMI constraint and linear objective function. Finally, the LMI toolbox in MATLAB is used to solve A_c, B_c, C_c and D_c.

4.Simulation example

In this section, the single inverted pendulum control system is used as a simulation model to verify the effectiveness of the designed H_∞ controller. For the mathematical model of single inverted pendulum control system [11], the system state space equation with external force as input is as follows.

$$\begin{bmatrix}\dot{x}\\ \ddot{x}\\ \dot{\varphi}\\ \ddot{\varphi}\end{bmatrix}=\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & A_{22} & m^2gl^2\Xi & -mlf\Xi\\ 0 & 0 & 0 & 1\\ 0 & -mlb\Xi & A_{43} & A_{44}\end{bmatrix}\begin{bmatrix}x\\ \dot{x}\\ \varphi\\ \dot{\varphi}\end{bmatrix}+\begin{bmatrix}0\\ ml\Xi\\ 0\\ B_{\omega 4}\end{bmatrix}\omega+\begin{bmatrix}0\\ B_{u2}\\ 0\\ ml\Xi\end{bmatrix}u$$

where, $A_{22}=-\left(I+ml^2\right)b\Xi$, $A_{43}=mgl\left(M+m\right)\Xi$,

$$y=\begin{bmatrix}x\\ \varphi\end{bmatrix}=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}x\\ \dot{x}\\ \varphi\\ \dot{\varphi}\end{bmatrix}+\begin{bmatrix}0\\ 0\end{bmatrix}\omega+\begin{bmatrix}0\\ 0\end{bmatrix}u$$

$A_{44}=\left(M-m\right)f\Xi$, $B_{\omega 4}=\left(M+m\right)\Xi$, $B_{u2}=\left(I+ml^2\right)\Xi$, $\Xi=1\Big/\left(M+m\right)I+Mml^2$.

Model parameters of inverted pendulum: The quality of the car $M=1.096kg$,The quality of the pendulum rod $m=0.109kg$,Friction coefficient of the car $b=0.1N/m/sec$,The length of the pendulum rod $l=0.25m$,Inertia of the pendulum rod $I=0.0034kg\cdot m^2$,With the prescribed parameters, equation (16) can be solved by using the Matlab LMI toolbox.

$$A_c=10^4\times\begin{bmatrix}0.0605 & 0.2379 & 0.4777 & -1.4968\\ 0.0319 & 0.1235 & 0.2504 & -0.7861\\ -0.0717 & -0.2736 & -0.5616 & 1.7691\\ -0.0151 & -0.0583 & -0.1183 & 0.3713\end{bmatrix},\ B_c=10^3\times\begin{bmatrix}0.0013 & -1.2441\\ 0.0006 & -0.6519\\ -0.0012 & 1.4608\\ -0.0002 & 0.3079\end{bmatrix},$$

$C_c=\begin{bmatrix}-63.6298 & -71.9102 & -204.0240 & 230.0706\end{bmatrix}$,

$D_c=\begin{bmatrix}2.4682 & 12.2183\end{bmatrix}$.

Fig. 1 shows the step response waveform of a single inverted pendulum system under the action of the designed H controller in the absence of external random interference. Then Figure 2 shows the step response waveform in the case of external random interference. The simulation results show that the displacement of the car and the swing angle of the swing rod can achieve stable state in 15s. That is to say, the H_∞controller designed by this method is effective and has good robustness and stability for the control of inverted pendulum system.

5.Conclusion

The design problem of H_∞ controller has been considered in this paper for a class single inverted pendulum based on output feedback. The design problem of the H_∞ controller has been transformed into a class of convex optimization problems with LMI constraints and linear objective functions by using optimization techniques. A sufficient condition for the existence of the controller has been established by means of LMI, and the parameters of the controller are determined directly by solutions of the LMI. The simulation result has showed that the controller

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designed in this paper has good effect on maintaining the balance stability of single inverted pendulum and has good anti-interference ability.

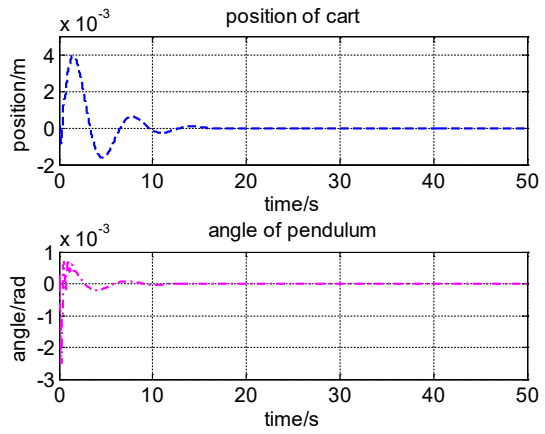

Fig.1 Step response curve of single inverted pendulum without interference

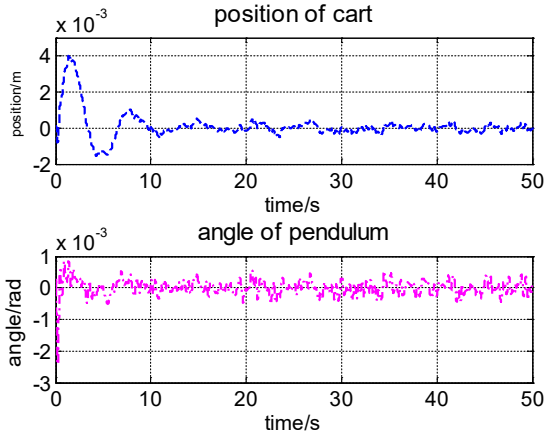

Fig.2 Step response curve of single inverted pendulum with interference

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