Journal of Computational Science & Engineering 29 (2017) 814-818

Available at http://www.asocse.org ISSN 1710-4068

Confidence intervals for process capability indices under unbalanced one-way random effect model

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Abstract: **The generalized confidence intervals for the process capability indices (PCIs) are given in the unbalanced one-way random effect model using fiducial inference. The simulation studies show that the performance of fiducial generalized inference is obviously close to the nominal confidence level. An industrial example is provided to illustrate the results.**

Keywords: Process capability indices; Generalized pivotal quantity; Fiducial generalized inference; Unbalanced one-way random effect model

Received 28 March 2017, Revised 23 May 2017, Accepted 25 May 2017

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1. Introduction

The process capability indices are widely used in manufacturing industries to provide a quantitative measurement of the performance of the products. It can reflect the influence of the system factors and the random factors on the production process. For the study of the process capability indices of the normal distribution of the simple case, A large number of domestic and foreign statistical scholars have discussed, see e.g., Kane[1] , Kotzand Johnson[2], Kotz and Lovelace [3], Kotz and Norman[4]. The most commonly usedprocess capability indices (PCIs) is C_{pk} , which is given as follows:

$$C_{pk}=\frac{\min\{USL-\mu,\mu-LSL\}}{3\sigma}=\frac{d-|\mu-M|}{3\sigma}, \qquad (1.1)$$

where μ is the process mean , σ is the process standard deviation, USL is the upper specification limit, LSL is the lower specification limit, and T is the target value, $d=(USL-LSL)/2$ and $M=(USL+LSL)/2$ is the specification center. C_{pk} is used to evaluate the capability of 'centered' processes , whereas the others are for 'off-centered' processes.

There are several papers works on confidence interval of the process capability indices under the balanced one-way random effect model, see Daniels and Burdick[5], Kurian and Mathew[6], and Jose and Luke[7]. However, most of the data obtained from the experiments are unbalanced due to the limitation of experimental conditions in practical applications. However, the exist methods do not work well under unbalanced cases.

In this paper, we give a new approach for confidence intervals of the process capability indices are obtained under unbalanced case by the method of fiducial inference. In Section 2, we introduce the one-way random effect model under unbalanced data. The concept of the generalized pivotal quantities was introduced to construct generalized confidence intervals and fiducial generalized confidence intervals for one-way random effect model under unbalanced data in Section 3. Section 4 present simulation study numerical results on the coverage rate and average length of of the confidence intervals. Illustrative example is given in Section 5.

2. The one-way random effect model with unbalanced data

Consider an experiment involving a single factor have a levels, with b_i observations corresponding to the ith level of the factor. The mathematical model for describing the relationship is given by

$$Y_{ij}=\mu+\alpha_i+\varepsilon_{ij}, i=1,2,\cdots,a; j=1,2,\cdots,b_i, \qquad (2.1)$$

where the μ is the common effect for the experiment, α_i represents the ith treatment effect and ε_{ij}'s represents the random error in the jth observation on the ith treatment effect. It assumed that α_i's and ε_{ij}'s are mutually independent normal random variables, with follow $N\left(0,\sigma_\alpha{}^2\right)$ and $N\left(0,\sigma_\varepsilon{}^2\right)$, respectively. Then the variance of Y_{ij} is $\sigma^2=\sigma_\alpha{}^2+\sigma_\varepsilon{}^2$, and the process capability indices (PCIs) for this model is

$$C_{pk}=\frac{d-|\mu-M|}{3\sigma}=\frac{d-|\mu-M|}{3\sqrt{\sigma_\alpha{}^2+\sigma_\varepsilon{}^2}} \qquad (2.2)$$

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We using the following notations

$$n=\sum_{i=1}^{a}b_i, n_1=a-1, n_2=n-a, b_h=\frac{a}{\sum_{i=1}^{a}1/b_i}, \bar{Y}_{i\cdot}=\sum_{j=1}^{b_i}Y_{ij}/b_i, \bar{Y}=(\bar{Y}_{1\cdot},\bar{Y}_{2\cdot},\cdots,\bar{Y}_{a\cdot})',$$

$$\bar{Y}_{\cdot\cdot}^{*}=\sum_{i=1}^{a}\bar{Y}_{i\cdot}/a, SS\bar{Y}=\sum_{i=1}^{a}b_i(\bar{Y}_{i\cdot}-\bar{Y}_{\cdot\cdot}^{*})^2 \text{ and } SSE=\sum_{i=1}^{a}\sum_{j=1}^{b_i}(Y_{ij}-\bar{Y}_{i\cdot})^2.$$

It is easy to know that

$$\bar{Y}\sim N\left(\mu 1_a, \sigma_\alpha^2 I_a+\sigma_\varepsilon^2 D\right), \text{and } SSE\sim\sigma_\varepsilon^2\chi_{n_2}^2 \text{ are independent,} \qquad (2.3)$$

where $D=diag(\frac{1}{b_1},\frac{1}{b_2},\cdots,\frac{1}{b_a})$.

In the unbalanced case, $SS\bar{Y}$ and SSE are still independent of each other, but $SS\bar{Y}$ is no longer subject to the chi square distribution. According to Thomas and Hulquist[8], $SS\bar{Y}\sim(\sigma_\alpha^2+\sigma_\varepsilon^2 b_h)\chi_{n_1}^2$ approximately.

It is easy to see $\bar{Y}_{\cdot\cdot}^{*}=\frac{1}{a}1'\bar{Y}$ and

$$\bar{Y}_{\cdot\cdot}^{*}\sim N(\mu,\frac{\sigma_\alpha^2+\sigma_\varepsilon^2/b_h}{a}) \qquad (2.4)$$

Let H be an $n_1\times a$ matrix such that $HH'=I_a-\frac{1}{a}J_a$, $HH'=I_{n_1}$, here J_a is an $a\times a$ matrix of ones. Let $B=HDH'$, then we have

$$H\bar{Y}\sim N(0,\sigma_\alpha^2 I_{n_1}+\sigma_\varepsilon^2 B) \qquad (2.5)$$

3. Generalized Confidence Intervals for C_{pk}

3.1. GCIs by Jose et al.[9]

Tsui and Weerahandi[10] proposed the concept of generalized test variable and generalized $p-value$. Later, Weerahandi[11] also extended to take advantage of the generalized pivotal quantities to construct generalized confidence intervals. Assumed an observable random vector X with a probability distribution $P_\eta(.)$, $\eta=(\theta,\delta)$ is an unknown parameter in the parameter space Ω, $\theta=\theta(\eta)$ is a real valued parameter function of interest, and δ is a nuisance parameter, and x is the observed value of the X .

Proposition 1. (see Weerahandi[11]) Let $R=R(X;x,\eta)$ be a function of X , the observed value x and the parameters $\eta=(\theta,\delta)$. R is said to be a generalized pivotal quantity if it has the following two properties:

(i) R has a probability distribution free of unknown parameters.

(ii)The observed value of R does not depend on the nuisance parameter .

Given a generalized pivotal quantity (GPQ) $R=R(X;x,\eta)$ and a confidence coefficient γ , let the subset of the sample space of C_γ be defined such that $P_\eta(R\in C_\gamma)=\gamma.$ Defined the subset Θ_c of the parameter space given by $\Theta_c(\gamma)=\{\theta\in\Theta: r(x,x,\eta)\in C_\gamma\}$. As a $100(1-\gamma)\%$ generalized confidence interval for R .

Let $\eta=(\mu,\sigma_\alpha^2,\sigma_\varepsilon^2)$ to derive a GPQ for C_{pk} in (2.1), we have to obtain a GPQ for μ and σ^2 respectively. Let $\bar{y}_{\cdot\cdot}^{*}$, $ss\bar{y}$ and sse denote the observed values of $\bar{Y}_{\cdot\cdot}^{*}$, $SS\bar{Y}$ and SSE . Jose et al.[9] provided the GPQ for μ and σ^2 as follows

$$R_\mu=\bar{y}_{\cdot\cdot}^{*}-\frac{Z}{U}\sqrt{\frac{ss\bar{y}}{a}}$$

$$R_\sigma=\frac{ss\bar{y}}{U^2}+(1-\frac{1}{b_h})\frac{sse}{V},$$

where $Z\sim N(0,1)$, $U^2\sim\chi_{n_1}^2$ and $V^2\sim\chi_{n_2}^2$ are mutually independent. Then the GPQ for C_{pk} in (2.1) given by

$$R_{C_{pk}}=\frac{d-|R_\mu-M|}{3R_\sigma} \qquad (3.1)$$

The generalized confidence intervals for C_{pk} can be constructed base on the distribution of $R_{C_{pk}}$ .For the observation y of Y and given $\xi\in(0,1)$, $R_{C_{pk}}(y;\xi)$ is the ξ percentile of the distribution of $R_{C_{pk}}$. Then the $(1-\gamma)\%$ equal tailed interval is

$$R_{C_{pk}}(y;\gamma/2), R_{C_{pk}}(y;1-\gamma/2) \qquad (3.2),$$

we refer this interval as the generalized confidence intervals $(GCIs)$.

3.2. The proposed method

Since $SS\bar{Y}$ is no longer subject to the chi square

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distribution (Thomas and Hulquist[8]), the GPQ in (2.1) is an approximately GPQ. The frequency coverage of the generalized confidence interval based on it could not maintain the stead level, especially for very unbalanced deigns. Li and Li[12] provided a fiducial method for constructing a generalized confidence interval on the sum of variance components, here we propose to use the concept of fiducial inference to construct the interval confidence for C_{pk} in a similar way. More details for derivation of fiducial distributions can be found in Hannig et al.[13].

Let $\mathbf{X}$ be a random vector from a probability distribution $\mathbf{P}_{\boldsymbol{\eta}}(\square)$, where η is an unknown vector, and $\theta = \theta(\zeta)$ is a parameter of interest. An observation from $\mathbf{X}$ is denoted by $\mathbf{x}$. Suppose that we have a structural equation, which can be expressed as

$$\mathbf{X} = \mathbf{G}(\boldsymbol{\zeta}, \mathbf{U})$$

for a random variable $\mathbf{U}$, whose distribution is known. Assume the equation $\mathbf{x} = \mathbf{G}(\boldsymbol{\zeta}, \mathbf{u})$ has a unique solution for ζ, say $\mathbf{G}^{-1}(\mathbf{x}, \mathbf{u})$. According to Hannig[13], the generalized fiducial quantity for ζ is

$$T_{\zeta} = \mathbf{G}^{-1}(\mathbf{x}, \mathbf{U})$$

and the generalized fiducial quantity for $\theta(\zeta)$ is

$$T_{\theta} = \theta(\mathbf{G}^{-1}(\mathbf{x}, \mathbf{U}))$$

.

Let $Z \sim N(0,1)$, $U_1 \sim N(0, I_{n_1})$, and $U_2 \sim \chi^2_{n_2}$ are mutually independent. By (2.3), (2.4) and (2.5), we have

$$\bar{Y}^*_{\cdot\cdot} = \mu + Z\sqrt{\frac{1}{a}(\sigma_\alpha^2 + \sigma_\varepsilon^2 / b_h)}, H\bar{Y} = (\sigma_\alpha^2 I_{n_1} + \sigma_\varepsilon^2 B)^{1/2} U_1, SSE = \sigma_\varepsilon^2 U_2 \qquad (3.3)$$

Since $\bar{Y}'H'H\bar{Y} = \bar{Y}'(I_a - \frac{1}{a}J_a)\bar{Y} = SS\bar{Y}$, we could obtain the following equation from (3.3)

$$SS\bar{Y} = \sigma_\alpha^2 U_1' U_1 + \sigma_\varepsilon^2 U_1' B U_1 \qquad (3.4)$$

Then by (2.3) and (2.4), the GPQ for μ and σ^2 could be proposed by the fiducial inference.

Lemma 1. Given the observation of y_{ij} of Y_{ij}. Define

$$T_\mu = T_\mu(y; Y, \eta) = \bar{y}^*_{\cdot\cdot} - \frac{Z}{\sqrt{a}}\sqrt{\frac{ss\bar{y}}{U_1'U_1} + (\frac{1}{b_h} - \frac{U_1'BU_1}{U_1'U_1})\frac{sse}{U_2}}$$

$$T_{\sigma^2} = T_{\sigma^2}(y; Y, \eta) = \frac{ss\bar{y}}{U_1'U_1} + (1 - \frac{U_1'BU_1}{U_1'U_1})\frac{sse}{W_2}$$

Then T_μ and T_{σ^2} are the generalized pivotal quantity μ and σ^2, respectively. Furthermore, $T_\mu(y; Y, \eta) = \mu, T_{\sigma^2}(y; Y, \eta) = \sigma^2$.

The proof of Lemma 1 is similar to that in Li and Li[12].

By Lemma 1, the generalized pivotal quantity $C_{pk} = \frac{d - |\mu - M|}{3\sqrt{\sigma_\alpha^2 + \sigma_\varepsilon^2}}$ could be given as follows,

$$T_{C_{pk}} = \frac{d - |T_\mu - M|}{3\sqrt{T_{\sigma^2}}} \qquad (3.5)$$

We can see that the observation of T is C_{pk}, the distribution of T is independent of the unknown parameter. So T satisfies the fiducial conditions. Note that the distribution of (Z, U_1, U_2) is completely known and absolute continuous. By Lemma 1, the distribution of $T_{C_{pk}}$ is free of unknown parameters. Hence generalized confidence intervals for C_{pk} can be constructed base on the distribution of $T_{C_{pk}}$. For the observation y of Y and given $\xi \in (0,1)$, $T_{C_{pk}}(y; \xi)$ is the ξ percentile of the distribution of $T_{C_{pk}}$. Then the $(1-\gamma)\%$ equal tailed generalized confidence interval is

$$[T_{C_{pk}}(y; \gamma/2), T_{C_{pk}}(y; 1-\gamma/2)] \qquad (3.6)$$

,

we refer this interval as the fiducial generalized confidence intervals $(FGCIs)$.

4. Simulation

Consider a single population in the one-way random effect model, we compare the fiducial generalized confidence intervals $(FGCIs)$ and the generalized confidence intervals $(GCIs)$. Let $\rho = \sigma_\alpha^2 / \sigma^2$ and $\sigma^2 = \sigma_\alpha^2 + \sigma_\varepsilon^2$. For the simulation, we chose $L = 7$, $U = 14$, $\mu = 10$, and $\sigma^2 = 1$. The coverage probabilities and the expected lengths of confidence intervals were computed for each design under different sample size. Take the confidence coefficient $1-\gamma = 95\%$, the values of ρ are selected to be 0.01, 0.05, 0.1, 0.2, 0.3, 0.5, 0.7 and 0.9, respectively. In this study, eight unbalanced patterns

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were selected for investigation, which are shown in Table 1. These observed values were generated 5,000 times for each design under different sample size using R languages.

Table1 unbalanced data

models	a	b_i
1	3	2 2 100
2	3	2 100 100
3	6	2 100 100 100 100 100
4	6	2 2 100 100 100 100
5	10	1 1 4 4 6 6 8 8 10 10
6	10	5 10 15 20 25 30 35 40 45 50
7	12	3 15 30 14 2 3 13 22 8 6 9 11
8	12	13 40 7 14 22 30 3 2 12 2 21 4

The simulation shows that as follows:

(1).The proposed FGCIs method provides excellent coverage probabilities cross all values of ρ for all patterns. However, the simulated confidence coefficients by $GCIs$ method drop to unacceptable levels when ρ is small in the very unbalanced designs, say for designs 2, 3, 4, 5, 7 and 8.

(2). The results indicate that when $FGCIs$ and $GCIs$ method perform well, i.e., for ρ is large, the average interval lengths of these two methods differ little.

In summary, comparing to $GCIs$, $FGCIs$ has good properties of the frequency. It can guarantee the coverage of the interval estimation, especially for the model with very unbalanced. Meanwhile, $GCIs$ method is affected by three kinds of conditions.

Table 2 The coverage and average length of the confidence intervals

		ρ							
model	method	0.01	0.05	0.1	0.2	0.3	0.5	0.7	0.9
1	*GCIs*	0.919 (0.990)	0.924 (1.020)	0.933 (1.055)	0.930 (1.115)	0.927 (1.182)	0.931 (1.330)	0.938 (1.555)	0.942 (2.011)
	FGCIs	0.941 (1.048)	0.943 (1.077)	0.945 (1.109)	0.939 (1.167)	0.938 (1.231)	0.933 (1.372)	0.941 (1.592)	0.942 (2.036)
2	*GCIs*	0.915 (0.745)	0.915 (0.807)	0.920 (0.874)	0.930 (0.967)	0.926 (1.049)	0.941 (1.222)	0.932 (1.444)	0.931 (1.901)
	FGCIs	0.940 (0.801)	0.939 (0.856)	0.938 (0.918)	0.944 (1.006)	0.936 (1.087)	0.941 (1.254)	0.934 (1.472)	0.932 (1.921)
3	*GCIs*	0.908 (0.284)	0.924 (0.347)	0.923 (0.411)	0.932 (0.519)	0.934 (0.610)	0.944 (0.781)	0.946 (0.965)	0.952 (1.238)
	FGCIs	0.960 (0.364)	0.957 (0.407)	0.952 (0.457)	0.951 (0.554)	0.942 (0.638)	0.946 (0.798)	0.948 (0.975)	0.953 (1.240)
4	*GCIs*	0.886 (0.397)	0.912 (0.445)	0.916 (0.495)	0.931 (0.580)	0.943 (0.665)	0.947 (0.819)	0.944 (0.996)	0.945 (1.254)
	FGCIs	0.950 (0.482)	0.954 (0.515)	0.951 (0.555)	0.950 (0.628)	0.954 (0.705)	0.951 (0.846)	0.946 (1.011)	0.946 (1.257)
5	*GCIs*	0.923 (0.511)	0.928 (0.521)	0.934 (0.540)	0.934 (0.580)	0.945 (0.621)	0.946 (0.713)	0.952 (0.817)	0.945 (0.963)
	FGCIs	0.961 (0.626)	0.961 (0.628)	0.959 (0.637)	0.953 (0.657)	0.958 (0.657)	0.953 (0.750)	0.954 (0.834)	0.944 (0.966)
6	*GCIs*	0.954 (0.219)	0.953 (0.252)	0.952 (0.292)	0.944 (0.367)	0.949 (0.442)	0.948 (0.581)	0.952 (0.733)	0.950 (0.931)
	FGCIs	0.956 (0.223)	0.955 (0.255)	0.955 (0.294)	0.946 (0.369)	0.951 (0.443)	0.949 (0.582)	0.950 (0.734)	0.950 (0.931)
7	*GCIs*	0.947 (0.318)	0.938 (0.337)	0.945 (0.362)	0.945 (0.414)	0.948 (0.468)	0.944 (0.573)	0.948 (0.697)	0.955 (0.851)
	FGCIs	0.960 (0.335)	0.948 (0.353)	0.954 (0.375)	0.950 (0.424)	0.952 (0.475)	0.945 (0.578)	0.949 (0.699)	0.954 (0.851)
8	*GCIs*	0.933 (0.310)	0.943 (0.331)	0.940 (0.358)	0.947 (0.412)	0.947 (0.465)	0.947 (0.574)	0.950 (0.697)	0.948 (0.855)
	FGCIs	0.954 (0.337)	0.957 (0.354)	0.955 (0.378)	0.954 (0.427)	0.952 (0.477)	0.950 (0.581)	0.950 (0.700)	0.949 (0.856)

5. An industrial example

To further illustrate the theoretical conclusions, we use an example taken from Montgomery [14]. There are 25 batches data resulting in a one-way model with random batch effects. We make the data become unbalanced by randomly deleting some of the data. Afterwards, the resulting sizes of the 25 batches are2,5,5,4,5,3,5,2,5,4,5,5,5,3, 5,5,5,2,5,4,5,3,5,4,5. And, the specification upper limits is 73.95, the specification lower limits is 74.05. Then, the observed values $\bar{y}^* = 74.001198$, $SS\bar{Y}^* = 0.000768$, and $SSE^* = 0.0078952$. Therefore, we can get $\hat{C}_{pk} = 1.8136$. About the confidence interval by using the Jose'method is (1.5412, 2.0568), and the confidence interval by the fiducial inference is (1.5408, 2.070). The Jose' method is little liberal than fiducial method.

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