Journal of Computational Science & Engineering 23 (2016) 710-713

Available at http://www.asocse.org ISSN 1710-4068

Matrix Representation of Solution Concepts and application in Multiple-Decision-Maker Graph Models

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Abstract: **Explicit matrix representations of solution concepts in a graph model of a multiple-decision-maker conflict with preference uncertainty are developed. In this paper, Nash stable is denoted by dot product. And the stable states for all DMs are formulated by matrix representation. At last, this paper analyses the conflict between production study and research alliance, showing the effectiveness and operability of the matrix representation.**

Keywords: Graph model; Matrix representation; Solution concepts; Stability analysis

Received 25 February 2016, Revised 23 March 2016, Accepted 25 March 2016

1. Introduction

A strategic conflict, a situation in which two or more decision makers (DMs) with different objectives interact, is a common occurrence in the real world. The graph model provides a methodology for modeling and analyzing strategic conflicts. Details of the definitions of the four basic stabilities of the graph model are in [1]. Xu defined the matrix representation of the four solution concepts for two DMs in [2]. Xu used matrices formulated a graph model and four graph model solution concepts for *n*-DM situation [3].

In this paper, we define the $m \times m$ matrix $M_i^{\mathrm{Nash}_n}$ to calculate Nash stable and define M^{K_n} to calculate stable state for all DMs.

2. Methods

To discuss the representation of a graph model using matrices, we first introduce some notations.

2.1. The Graph Model for Conflict Resolution

A graph model for a strategic conflict is comprised of a finite set of DMs N, a set of feasible states S, a preference relation $\succeq_i$ on S for each DM i, and a directed graph $G_i = \{S, A_i\}$. In each directed graph, S is the vertex set, and each oriented arc of the arc set $A_i \subseteq S \times S$ indicates that DM i can make a legal move (in one step) from the initial state to the terminal state of the arc.

For $i \in N$, J_i is a $|S| \times |S|$ 0-1 matrix defined by

$$J_i(s,q) = \begin{cases} 1 & \text{if } (s,q) \in A_i, \\ 0 & \text{otherwise,} \end{cases}$$

where |S| is the number of the states in S. J_i is called a reachability matrix to represent DM i's unilateral moves (UMs). $R_i(s)$ denotes DM i's reachable list from a state s, containing all states to which DM i can move from state s in one step. So $R_i(s) = \{q : J_i(s,q) = 1\}$. If $R_i(s)$ is written as a 0–1 row vector, then $R_i(s) = e_s^T \cdot J_i$ where e_s^T denotes the transpose of the sth standard basis vector of the m-dimensional Euclidean space. For DM i, a unilateral improvement (UI) matrix J_i^+ is defined as follows:

$$J_i^+(s,q) = \begin{cases} 1 & \text{if } J_i(s,q) = 1 \text{ and } q \succ_i s, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, $R_i^+(s)$ is defined as DM i's UIs from state s. Therefore, $R_i^+(s) = \{q : J_i^+(s,q) = 1\}$ .It is clear that $R_i^+(s) = e_s^T \cdot J_i^+$ if $R_i^+(s)$ is written as a 0–1 row vector.

2.2. Preference Relation Matrix

Let E represents an |S| × |S| matrix with each entry 1, and let e_k denote a |S|−dimensional column vector with k_{th} element 1 and all other elements 0, let e denote a |S|−dimensional column vector with all elements 1. For two |S| × |S| matrices M and N, $W = M \circ N$ is defined as the |S|×|S| matrix with (s,q) entry W(s,q) = M(s,q) · N(s,q). (" ∘ " denotes the Hadamard product.) If M is an |S|×|S| matrix, then the |S|×|S| matrix sign(M) with (s,q) entry is defined by

$$sign[m(s,q)] = \begin{cases} 1 & m(s,q) > 0, \\ 0 & m(s,q) = 0, \\ -1 & m(s,q) < 0. \end{cases}$$

Preference relation matrices P_i^+ for DM i is respectively defined as $P_i^+(s,q) = \begin{cases} 1 & \text{if } q \succ_i s, \\ 0 & \text{otherwise.} \end{cases}$

Then matrix $P_i^{-,=}$ with (s,q) entry is defined by

$$P_i^{-,=}(s,q) = \begin{cases} 1 - P_i^+(s,q), & \text{if } s \neq q, \\ 0 & \text{otherwise.} \end{cases} \quad \text{and}$$

$$P_i^{-,=}(s,q) = [P_i^+(s,q)]^T .$$

Xu researched the matrix representation of the four solution concepts for n-DM situation in [18].But Nash

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stable is denoted by inner product. So we define the Nash stable by dot product as follow.

A state s∈S is Nash stable for DM i iff i's UI list from s, $R_i^+(s)$, is empty. Define the $m\times m$ matrix $M_i^{\text{Nash}_n}$ by $M_i^{\text{Nash}_n} = J_i^+ \cdot E$

Theorem 1: Let i∈N and |N|=n. A state s∈S is Nash stable for DM i, denoted by $s \in S_i^{\text{Nash}_n}$, iff $M_i^{\text{Nash}_n}(s,s)=0$.

For GMR, DM i will take into account the opponents' possible responses, which are legal sequences of UMs by members of the subset $R_{N-\{i\}}$. W=M∨N is defined as the m×m matrix with (s,q) entry W(s,q) expressed as

$$W(s,q)=\begin{cases}1, & \text{if } M(s,q)+N(s,q)\neq 0\\ 0, & \text{otherwise.}\end{cases}$$

Let $i \in N$ with |N|=n and find matrix $M_{N-\{i\}}$ using $M_H = \overset{\delta}{\underset{t=1}{\bigvee}}\bigvee_{i\in H} M_i^{(t)}$, for which H=N−{i} and

$$\delta = e^{\mathrm{T}} \cdot J_i \cdot e$$

Define the m×m matrix $M_i^{\text{GMR}_n}$ by $M_i^{\text{GMR}_n} = J_i^+ \cdot [E - \text{sign}(M_{N-\{i\}} \cdot P_i^+)]$

Theorem 2: Let $i \in N$ and |N|=n. A state s∈S is general metarational for DM i, denoted by $s \in S_i^{\text{GMR}_n}$, iff $M_i^{\text{GMR}_n}(s,s)=0$.

Table 1 Options and feasible states for the conflict

Enterprise												
1.fully-collaborative	Y	N	N	Y	N	N	Y	N	N	Y	N	N
2.cooperate partly	N	Y	N	N	Y	N	N	Y	N	N	Y	N
3.opportunism	N	N	Y	N	N	Y	N	N	Y	N	N	Y
College												
4.cooperate	N	N	N	Y	Y	Y	N	N	N	Y	Y	Y
5.opportunism	Y	Y	Y	N	N	N	Y	Y	Y	N	N	N
Government												
6.encourage	Y	Y	Y	Y	Y	Y	N	N	N	N	N	N
7.discourage	N	N	N	N	N	N	Y	Y	Y	Y	Y	Y
State number	1	2	3	4	5	6	7	8	9	11	11	12

For SMR, the n-DM model is similar to the 2-DM model. The only modification is that responses come from DM i's opponents instead of from a single DM. Let $i \in N$ and |N| = n. Let

$$W = P_i^+ \circ [E - \text{sign}(J_i \cdot (P_i^+)^{\mathrm{T}})]$$

then define the m × m matrix $M_i^{\text{SMR}_n}$ by $M_i^{\text{SMR}_n} = J_i^+ \cdot [E - \text{sign}(M_{N-\{i\}} \cdot W)]$

Theorem 3: Let $i \in N$ and |N|=n. A state s∈S is symmetric metarational for DM i, denoted by $s \in S_i^{\text{SMR}_n}$, iff $M_i^{\text{SMR}_n}(s,s)=0$.

SEQ examines the credibility of the sanctions by DM i's opponents. Let i∈N with |N|=n and find matrix $M_{N-\{i\}}^+$ using $M_H^+ = \overset{\delta_1}{\underset{t=1}{\bigvee}}\bigvee_{i\in H} M_i^{(t,+)}$. Define the m×m matrix $M_i^{\text{SEQ}_n}$ by $M_i^{\text{SEQ}_n} = J_i^+ \cdot [E - \text{sign}(M_{N-\{i\}}^+ \cdot P_i^+)]$

Theorem 4: Let $i \in N$ and |N|=n. A state s∈S is sequentially stable for DM i, denoted by $s \in S_i^{\text{SEQ}_n}$, iff $M_i^{\text{SEQ}_n}(s,s)=0$.

Theorem 5: Define K stable is one of the four stable. Let $i \in N$ and |N|=n. A state s∈S is K stable for all DMs, denoted by $s \in S^{\text{K}_n}$, iff

$$M^{\text{K}_n}(s,s) = \sum_{i=1}^{n} M_i^{\text{K}_n}(s,s) = 0 \quad (\text{K} \in (\text{Nash}, \text{GMR}, \text{SMR}, \text{SEQ}))$$

This means stats s is K stable for all DMs. Therefore, we conclude the previous discussion.

We define the four stable matrixes to calculate the four stables based on UI matrix J_i^+ from theorem 1 to theorem 4. The matrix expression of the global stable is given by theorem 5.

Table 2. Preference Information

Enterprise	5>11>4>10>6>12>2>8>3>9>1>7
College	4>10>5>11>1>2>7>8>3>9>6>12
Government	4>5>10>11>1>2>7>8>6>12>3>9

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3. Application

Table 3 State Transition Table

State	Enterpris R1	Colleg R2	Governme R3
1	2,3	4	
2		5	
3	2		
4	5		
5			
6	4,5	3	
7	8,9	10	1
8		11	2
9	8		3
10	11		4
11			5
12	10,11	9	6

The conflict between production study and research alliance in our country is worth studying. The conflict of production study and research alliance in our country is modeled by three DMs: DM 1, Enterprise (E), with three strategies: fully-collaborative (labeled F), cooperate partly (labeled P), opportunism (labeled O); DM 2, College (C), with two strategies: cooperate (labeled C), opportunism (labeled O); and DM 3, Government (G),with two strategies: encourage (labeled E),discourage (labeled D).

Let N = {1,2,3}, the set of three DMs. We use the production study and research alliance model as an example to show the matrix representation of the four solution concepts in the graph model obtained by carrying out the following steps.

1) Construct matrices J_i, J_i^+ and P_i^+ for i =1,2,3.

2) Calculate the joint movement and improvement matrices.

3) Analyze the stabilities of the conflict.

UM, UI, and Preference Relation Matrices:

$$J_1^+ = \begin{pmatrix} 0&1&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&1&0 \end{pmatrix} \quad J_2^+ = \begin{pmatrix} 0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&0&0 \end{pmatrix} \quad J_3^+ = \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0 \end{pmatrix}$$

$$P_1^+ = \begin{pmatrix} 0&1&1&1&1&1&0&1&1&1&1&1\\ 0&0&0&1&1&1&0&0&0&1&1&1\\ 0&1&0&1&1&1&0&1&0&1&1&1\\ 0&0&0&0&1&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&1&0&0&0&0&1&1&0\\ 1&1&1&1&1&1&0&1&1&1&1&1\\ 0&1&0&1&1&1&0&0&0&1&1&1\\ 0&1&1&1&1&1&0&1&0&1&1&1\\ 0&0&0&1&1&0&0&0&0&0&1&0\\ 0&0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&1&1&0&0&0&1&1&0 \end{pmatrix} \quad P_2^+ = \begin{pmatrix} 0&0&0&1&1&0&0&0&0&1&1&0\\ 1&0&0&1&1&0&0&0&0&1&1&0\\ 1&1&0&1&1&0&1&1&0&1&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&1&0&0\\ 1&1&1&1&1&0&1&1&1&1&1&0\\ 1&1&0&1&1&0&0&0&0&1&1&0\\ 1&1&0&1&1&0&1&0&0&1&1&0\\ 1&1&1&1&1&0&1&1&0&1&1&0\\ 0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&1&1&0&0&0&0&1&0&0\\ 1&1&1&1&1&1&1&1&1&1&1&0 \end{pmatrix} \quad P_3^+ = \begin{pmatrix} 0&0&0&1&1&0&0&0&0&1&1&0\\ 1&0&0&1&1&0&0&0&0&1&1&0\\ 1&1&0&1&1&1&1&1&0&1&1&1\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&1&0&0\\ 1&1&0&1&1&0&1&1&0&1&1&0\\ 1&1&0&1&1&0&0&0&0&1&1&0\\ 1&1&0&1&1&0&1&0&0&1&1&0\\ 1&1&1&1&1&1&1&1&0&1&1&1\\ 0&0&0&1&1&0&0&0&0&0&0&0\\ 0&0&0&1&1&0&0&0&0&1&0&0\\ 1&1&0&1&1&1&1&1&0&1&1&0 \end{pmatrix}$$

2) Joint Movement and Improvement Matrices:

Let N={1,2,3} and H=N−{i}, for i = 1,2,3.

$$M_H = \bigvee_{t=1}^{\delta} \bigvee_{i \in H} M_i^{(t)}$$

where

$$M_i^{(t)} = \text{sign}\left(J_i \left(\bigvee_{j \in H-\{i\}} M_j^{(t-1)} \right) \right)$$

with $M_i^{(1)} = J_i$ and $M_H^+ = \bigvee_{t=1}^{\delta} \bigvee_{i \in H} M_i^{(t,+)}$

where

$$M_i^{(t,+)} = \text{sign}\left(J_i \left(\bigvee_{j \in H-\{i\}} M_j^{(t-1,+)} \right) \right)$$

with $M_i^{(1,+)} = J_i^+$ and δ =24.

Now, the process to obtain each joint movement matrix M_H is shown as follows: Let H = N−{1}. Then

$$M_{N-\{1\}} = \bigvee_{t=1}^{\delta} (M_2^{(t)} \vee M_3^{(t)})$$

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where

$$M_2^{(t)} = \text{sign}(J_2 \cdot M_3^{(t-1)}), \quad M_3^{(t)} = \text{sign}(J_3 \cdot M_2^{(t-1)})$$

For i=2,3, joint movement matrices $M_{N-\{1\}}$ can be similarly calculated. Based on the aforementioned discussion, the construction of joint improvement matrix M_H^+ can be easily determined.

$$M^{\text{Nash}_3} = \sum_{i=1}^{3} M_i^{\text{Nash}_3} = \begin{pmatrix} 3&3&3&3&3&3&3&3&3&3&3&3 \\ 1&1&1&1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1&1&1&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 3&3&3&3&3&3&3&3&3&3&3&3 \\ 4&4&4&4&4&4&4&4&4&4&4&4 \\ 2&2&2&2&2&2&2&2&2&2&2&2 \\ 2&2&2&2&2&2&2&2&2&2&2&2 \\ 2&2&2&2&2&2&2&2&2&2&2&2 \\ 1&1&1&1&1&1&1&1&1&1&1&1 \\ 4&4&4&4&4&4&4&4&4&4&4&4 \end{pmatrix} \quad M^{\text{GMR}_3} = \bigvee_{i=1}^{3} M_i^{\text{GMR}_3} = \begin{pmatrix} 3&3&3&3&3&3&3&3&3&3&3&3 \\ 1&1&1&1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1&1&1&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 3&3&3&3&3&3&3&3&3&3&3&3 \\ 3&3&4&2&3&4&3&3&4&2&3&4 \\ 2&2&2&2&2&2&2&2&2&2&2&2 \\ 1&1&2&1&1&2&1&1&2&1&1&2 \\ 2&2&2&2&2&2&2&2&2&2&2&2 \\ 1&1&1&1&1&1&1&1&1&1&1&1 \\ 3&4&4&3&3&4&4&4&4&3&3&4 \end{pmatrix}$$

$$M^{\text{SMR}_3} = \bigvee_{i=1}^{3} M_i^{\text{SMR}_3} = \begin{pmatrix} 3&3&3&3&3&3&3&3&3&3&3&3 \\ 1&1&1&1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1&1&1&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 3&3&3&3&3&3&3&3&3&3&3&3 \\ 3&3&3&3&3&3&3&3&3&3&3&3 \\ 2&2&2&2&2&2&2&2&2&2&2&2 \\ 2&2&2&2&2&2&2&2&2&2&2&2 \\ 2&2&2&2&2&2&2&2&2&2&2&2 \\ 1&1&1&1&1&1&1&1&1&1&1&1 \\ 3&3&3&3&3&3&3&3&3&3&3&3 \end{pmatrix} \quad M^{\text{SEQ}_3} = \bigvee_{i=1}^{3} M_i^{\text{SEQ}_3} = \begin{pmatrix} 3&3&3&3&3&3&3&3&3&3&3&3 \\ 1&1&1&1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1&1&1&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 3&3&3&3&3&3&3&3&3&3&3&3 \\ 3&3&4&2&3&4&3&3&4&2&3&4 \\ 2&2&2&2&2&2&2&2&2&2&2&2 \\ 1&1&2&1&1&2&1&1&2&1&1&2 \\ 2&2&2&2&2&2&2&2&2&2&2&2 \\ 1&1&1&1&1&1&1&1&1&1&1&1 \\ 3&4&4&3&3&4&4&4&4&3&3&4 \end{pmatrix}$$

So state 5 is the Nash, GMR, SMR and SEQ stable for all DMs.

4. Conclusion

In this paper, the MRSC is developed to represent solution concepts in multiple-DM graph models. Matrix calculations of individual stability and equilibria are defined for the following solution concepts: Nash stability, GMR, SMR, and SEQ. In n-DM models, the joint UMs R_H and UIs R_H^+ are essential to solution concept definitions. Fang et al. in [5] provide a pseudo code for constructing R_H and R_H^+, a complicated process. Therefore, the idea of the joint UM matrix M_H and the joint UI matrix M_H^+ is significant.

In the previous reference, Nash stable is denoted by inner product. So we define

$$M_i^{\text{Nash}_n} = J_i^+ \cdot E$$

In order to calculate stable state for all DMs by matrix representation, we define

$$M^{\text{K}_n} = \sum_{i=1}^{n} M_i^{\text{K}_n} \qquad (\text{K} \in (\text{Nash}, \text{GMR}, \text{SMR}, \text{SEQ}))$$

At last, we analyze the conflict between production study and research alliance by matrix representation.

References

[1] L. Fang, K. W. Hipel, D. M. Kilgour, Interactive Decision Making: The Graph Model for Conflict Resolution, New York: Wiley, 1993.

[2] H. Xu, K.W. Hipel, D.M. Kilgour. IEEE Int. Conf. Syst., Man, Cybern.

[3] H. Xu, K.W. Hipel, D.M. Kilgour, IEEE Trans. Syst., Man, Cybern. A, Syst., Humans, 39 (2009) 96–108.

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