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Several Theorems Proving of Double Integrals of Function in the Polar Coordinate

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Abstract

In the calculation of double integral of function, a simple method is used not only simplifies the operation step, but also is easy to understand. For some special integration region and the integrand in the past, usually in order to achieve the purpose of simplifying integral functions and integration region by polar coordinate transformation; or some integration region and the integrand are directly represented by polar coordinates, we have adopted polar coordinate transformation method. In order to better calculate double integrals of function in polar coordinate, according to the characteristics of the polar coordinate system, this paper under the inspiration of the rectangular coordinate and the conclusions of this thesis, will further promote periodicity and symmetry of double integral of function in polar coordinates, periodic theorem and symmetry theorem proving are given.

1. Introduction

With the innovation and development of science and technology, the double integral of function is widely used in much geometry, physics, and other practical problems, in the calculation process often seek a more convenient way. At home and abroad the double integral of function has made great breakthroughs and achievements in the polar coordinate transformation, and the theoretical basis is further established for development of basic mathematics and applied mathematics. Many research results have certain practical guiding significance. In the learning of "Mathematical Analysis" textbook, when calculating double integral of function, for some special integration region and the integrand, often through the polar coordinate transformation to simplify the integration region and integral function, ultimately obtain the purpose of simplifying the calculation procedure. Therefore, the research of double integral of function in

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polar coordinates has great significance. But through the network, publications and other reference materials, there are not collected similar articles specifically for the research of double integral of function in polar coordinates.

When calculating the double integral of function, usually in the rectangular coordinate, the symmetry of integration region and parity features of integrand is adopted to calculate the double integral of function to meet integral theorem, so not only can simplify the calculus, but also can avoid errors; then in polar coordinates, we can get a similar conclusion? Meanwhile, in polar coordinates, considering periodic the existence the angle, then during the integration time, whether or not periodicity exists? This paper mainly focuses on the periodicity and symmetry, which discussed and given periodicity and symmetry theorem.

2. Preliminary knowledge

In "Mathematical Analysis" [4] textbooks, for definite integral of function, multiple integral of multi-function and other knowledge are comprehensively explained. This section mainly briefly describes the knowledge that the below needs.

Theorem 1[1]: if $f(x)$ is Cycle of continuous function as p in $(-\infty,+\infty)$, Then for any real number, there is a constant

$\int_{a+p}^{b+p} f(x)dx = \int_{a}^{b} f(x)dx = \int_{a+kp}^{b+kp} f(x)dx$, k is an integer

Theorem 2[4]: If the function group $u = u(x,y)$, $v = v(x,y)$ has continuous partial derivatives in an open area , and $\forall (x,y) \in G$, $J(x,y) \equiv \frac{\partial(u,v)}{\partial(x,y)} \neq 0$. Function group change open area G in the xy plane into open area G' in the uv plane. Point $(x_0,y_0) \in G$ transforms point $(u_0,v_0) = [u(x_0,y_0), v(x_0,y_0)] \in G'$ in the uv plane, then the area infinitesimal $d\sigma'$ of contained point (u_0,v_0) and the area infinitesimal $d\sigma'$ of corresponding contained points (x_0,y_0) is $|J(x_0,y_0)|$, that is

$$\frac{d\sigma'}{d\sigma} = |J(x_0,y_0)| = \left|\frac{\partial(u,v)}{\partial(x,y)}\right|_{(x_0,y_0)}$$

Theorem 3[4]: If the function $f(x,y)$ has R continuity in a closed bounded region, function group

$x = x(u,v)$, $y = y(u,v)$

region R' in the uv plane is transformed the region R' in the xy plane on by one. And the function group on the R' and on u has continuous partial derivatives, $\forall (u,v) \in R'$ $J = \frac{\partial(x,y)}{\partial(u,v)} \neq 0$, then

$$\iint_R f(x,y)dxdy = \iint_{R'} f[x(u,v), y(u,v)]|J(u,v)|dudv$$

3. Two main conclusions

3.1.Periodicity

In the Literature [1], double integral instance is calculated functions by using polar coordinate transformation has been given, and use the θ cycle to solve students' understanding of integration results. On this basis, this section discusses when the integrand $f(r\cos\theta, r\sin\theta)$ and integral boundary function $r_1(\theta)$ cycle is ω ($\omega \leq 2\pi$), integration period of double integral of planar region function in polar coordinates, the following will be given the corresponding proof.

Theorem 2.1 for double integral $\int_{\alpha}^{\beta} d\theta \int_{0}^{r_1(\theta)} f(r,\theta)dr = \int_{\alpha}^{\beta} [\int_{0}^{r_1(\theta)} f(r,\theta)dr]d\theta$, if the of cycle $f(r,\theta)$ and $r_1(\theta)$ on θ is ω ($\omega \leq 2\pi$),

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then(1)

$\int_{\alpha}^{\beta}[\int_{0}^{r_1(\theta)} f(r,\theta)dr]d\theta = \int_{\alpha+k\omega}^{\beta+k\omega}[\int_{0}^{r_1(\theta)} f(r,\theta)dr]d\theta$, k is an integer;

(2) If $k\omega = 2\pi$ and $[\alpha,\beta]$ is periodic interval of $f(r,\theta)$ effective integration, that is

$$\int_0^{2\pi}[\int_0^{r_1(\theta)} f(r,\theta)dr]d\theta=\int_0^{k\omega}[\int_0^{r_1(\theta)} f(r,\theta)dr]d\theta=k\int_{\alpha}^{\beta}[\int_0^{r_1(\theta)} f(r,\theta)dr]d\theta$$

Proof: (1) make $F(\theta)=\int_0^{r_1(\theta)} f(r,\theta)dr$, $f(r,\theta)$ and $r_1(\theta)$ about θ cycle is ω ($\omega \le 2\pi$) continuous function.

Obviously

$F(\theta)=\int_0^{r_1(\theta)} f(r,\theta)dr=\int_0^{r_1(\theta+\omega)} f(r,\theta+\omega)dr=F(\theta+\omega)$, and meet $\lim_{\Delta\theta\to 0}F(\theta+\Delta\theta)=\lim_{\Delta\theta\to 0}\int_0^{r_1(\theta+\Delta\theta)} f(r,\theta+\Delta\theta)dr=\int_0^{r_1(\theta)} f(r,\theta)dr=F(\theta)$, Then $F(\theta)$ is a continuous function which takes ω as cycle. The above Theorem 1 can obtain:

$$\int_{\alpha}^{\beta}[\int_0^{r_1(\theta)} f(r,\theta)dr]d\theta=\int_{\alpha}^{\beta}F(\theta)d\theta=\int_{\alpha+k\omega}^{\beta+k\omega}F(\theta)d\theta=\int_{\alpha+k\omega}^{\beta+k\omega}[\int_0^{r_1(\theta)} f(r,\theta)dr]d\theta$$

, k is an integer.

General polar coordinate transformation $\begin{cases}x=r\cos\theta\\ y=r\sin\theta\end{cases}$, $0\le r\le\infty$, $0\le\theta\le 2\pi$, For the boundary function $G(x,y)=0$, $G(r\cos\theta,r\sin\theta)=0$, $r_1(\theta)$ can be represented for the boundary function, and cycle is $\omega=2\pi$;For the integrand $f(x,y)$, that is $f(r\cos\theta,r\sin\theta)$, $f(x,y)$ Cycle is clearly $\omega=2\pi$ .and this time meet:

$$\int_{\alpha}^{\beta}[\int_0^{r_1(\theta)} f(r,\theta)dr]d\theta=\int_{\alpha}^{\beta}F(\theta)d\theta=\int_{\alpha+k\omega}^{\beta+k\omega}F(\theta)d\theta=\int_{\alpha+2k\pi}^{\beta+2k\pi}[\int_0^{r_1(\theta)} f(r,\theta)dr]d\theta$$

, k k is an integer.

(2) Because $f(r,\theta)$ and $r_1(\theta)$ about θ cycle is ω ($\omega\le 2\pi$, and $k\omega=2\pi$,then

$$\int_0^{2\pi}[\int_0^{r_1(\theta)} f(r,\theta)dr]d\theta=\int_0^{k\omega}[\int_0^{r_1(\theta)} f(r,\theta)dr]d\theta$$

And because $[\alpha,\beta]$ is $f(r,\theta)$ cycle interval of effective integration, by conclusion (1), then

$$\int_{\alpha}^{\beta}[\int_0^{r_1(\theta)} f(r,\theta)dr]d\theta=\int_{\alpha+\omega}^{\beta+\omega}[\int_0^{r_1(\theta)} f(r,\theta)dr]d\theta=\ldots=\int_{\alpha+(k-1)\omega}^{\beta+(k-1)\omega}[\int_0^{r_1(\theta)} f(r,\theta)dr]d\theta$$

for $f(r,\theta)$ in integration $[0,2\pi]$:

$$\int_0^{2\pi}[\int_0^{r_1(\theta)} f(r,\theta)dr]d\theta=\int_{\alpha}^{\beta}[\int_0^{r_1(\theta)} f(r,\theta)dr]d\theta+\int_{\alpha+\omega}^{\beta+\omega}[\int_0^{r_1(\theta)} f(r,\theta)dr]d\theta+\ldots+\int_{\alpha+(k-1)\omega}^{\beta+(k-1)\omega}[\int_0^{r_1(\theta)} f(r,\theta)dr]d\theta=k\int_{\alpha}^{\beta}[\int_0^{r_1(\theta)} f(r,\theta)dr]d\theta$$

k is an integer,

So the conclusion is clearly established

$$\int_0^{2\pi}[\int_0^{r_1(\theta)} f(r,\theta)dr]d\theta=\int_0^{k\omega}[\int_0^{r_1(\theta)} f(r,\theta)dr]d\theta=k\int_{\alpha}^{\beta}[\int_0^{r_1(\theta)} f(r,\theta)dr]d\theta$$

3.2. Symmetry

Symmetry has been widely used in the function integration

2.2.1 Regional D on the polar axis symmetry

Theorem 2.2.1 If the area D on the polar axis symmetry, $f(r,\theta)$ continue in D and $f(r,\theta)$ is θ singular or even function, then double integral

$$\iint_D f(r,\theta)drd\theta=\begin{cases}0, f(r,\theta)=-f(r,-\theta)\\ 2\iint_{D_1} f(r,\theta)drd\theta, f(r,\theta)=f(r,-\theta)\end{cases}$$

, where $D=D_1+D_2$, is the upper part of the polar axis

Proof: because $D=D_1+D_2$, it is known by the integration region

$$\iint_D f(r,\theta)drd\theta=\iint_{D_1} f(r,\theta)drd\theta+\iint_{D_2} f(r,\theta)drd\theta \quad (1)$$

Because D_1, D_2 is about symmetry, for any point $p(r,\theta)\in D_1$, its symmetry point

$p'(r,-\theta)\in D_2$, $D_1=\{0\le r\le r_1(\theta),\alpha\le\theta\le\beta\}$, $D_2=\{0\le r\le r_1(\theta),-\beta\le\theta\le-\alpha\}$.

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For, we make $\begin{cases} r = r \\ \theta = -\varphi \end{cases}$, by the above theorem [2–3], and D_2 change into $ro\varphi$ plane

$D_1 = \{0 \le r \le r_1(\theta), \alpha \le \varphi \le \beta\}$, where, Jacobian $\frac{\partial(r,\theta)}{\partial(r,\varphi)} = -1$.so

$$\iint_{D_2} f(r,\theta)drd\theta = \iint_{D_1} f(r,-\varphi) * |-1| drd\varphi = \iint_{D_1} f(r,-\theta)drd\theta$$

$$= \begin{cases} -\iint_{D_1} f(r,\theta)drd\theta, f(r,\theta) = -f(r,-\theta) \\ \iint_{D_1} f(r,\theta)drd\theta, f(r,\theta) = f(r,-\theta) \end{cases}$$

then formula(1)

$$\iint_D f(r,\theta)drd\theta = \iint_{D_1} f(r,\theta)drd\theta + \iint_{D_2} f(r,\theta)drd\theta$$

$$= \iint_{D_1} f(r,\theta)drd\theta + \begin{cases} -\iint_{D_1} f(r,\theta)drd\theta, f(r,\theta) = -f(r,-\theta) \\ \iint_{D_1} f(r,\theta)drd\theta, f(r,\theta) = f(r,-\theta) \end{cases}$$

$$= \begin{cases} 0, f(r,\theta) = -f(r,-\theta) \\ 2\iint_{D_1} f(r,\theta)drd\theta, f(r,\theta) = f(r,-\theta) \end{cases}$$

Therefore, the conclusion was founded

Similar to the following conclusions

Theorem 2.2.2, If the area D on the pole symmetry, $f(r,\theta)$ continue in D, then the double integral

$$\iint_D f(r,\theta)drd\theta = \begin{cases} 0, f(r,\theta) = -f(r,\pi+\theta) \\ 2\iint_{D_1} f(r,\theta)drd\theta, f(r,\theta) = f(r,\pi+\theta) \end{cases}$$

, where, (D_1 , D_2 on the pole symmetry).

Theorem 2.2.3: If area D on pole and perpendicular to the symmetry of axis line symmetry, $f(r,\theta)$ continue in D,.

Then double integral

$$\iint_D f(r,\theta)drd\theta = \begin{cases} 0, f(r,\theta) = -f(r,\pi-\theta) \\ 2\iint_{D_1} f(r,\theta)drd\theta, f(r,\theta) = f(r,\pi-\theta) \end{cases}$$

, where, $D = D_1 + D_2$,(D_1 , D_2 On pole and perpendicular to symmetry of straight line of the polar axis)

4. Conclusion

This paper discusses how to use polar coordinate to calculate double integral of function, for some symmetry or periodical regions that in polar coordinates plane, considering its integrand (for θ) parity or periodicity, using the conclusion of this paper can be changed be a hard job to easy, make hard things simple, make its reach twice the result with half the effort. So for Heavy integral calculation, it has great significance. At the same time, In the transformation of polar coordinates, the original variable x, y change into r, θ, because scope of the r is restricted, when considering θ separately, the periodicity, symmetry, the parity of integrand and so on, all simplifies the original binary operation procedures, this is also an advantage for using the polar coordinate.

Before there is a lot of research about symmetry in the calculation of function integral, there are very few in polar coordinates. This article focuses on the polar coordinate transformation; the integral calculation is given the corresponding conclusion, which further helps to solve and study related problems of calculation of the double integral functions.

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aid (Qian Education Science (2011) No. 039, 125 plan); Guizhou Minzu university fund projects 2013 "traffic sign detection and recognition based on computer vision".

References

[1] L.P. Ren, Journal of Lvliang College Journal, 3 (2009) 9-11.

[2] S.L. Jin, Y.Y. Fu, J.G. Chen, Journal of Cangzhou Normal Junior college, 9 (2011) 113-114.

[3] C.R. Xue, F. Wang, Science and Technology and Engineering, 1 (2010) 172-175.

[4] Y.L. Liu, P.R. Fu, D. Lin, Mathematical analysis (Beijing: higher education press, 2008).

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