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Classification of Entanglement for Triqutrit Mixed States with Bell Inequalities

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Abstract

Using Bell function and the generalized three dimensional Pauli operators, a set of Bell inequalities classifying the entanglement of triqutrit mixed states is presented. These inequalities can distinguish fully separable and bi-separable states. Moreover we present a quantum upper bound for triqutrit mixed states using generalized Bell function.

One of the most remarkable features that distinguish quantum mechanics from classical mechanics is entanglement. Entanglement was first recognized by EPR [1] and the significant progress toward the resolution of the EPR problem was made by Bell [2]. Since Bell's work, derivation of new inequalities is one of the important and challenging subjects. CHSH generalized the original Bell inequalities to a more general case for two observers [3]. For n partite systems with two dichotomic observable each, a general Bell inequality is constructed in Ref. [4]. An independent derivation of this inequality and its extensive classification was given by Werner and Wolf [5]. Still there have been many important generalizations and interesting applications for Bell inequalities [6, 7, 8]. A classification of various types of n-qubit entanglement was presented by Yu et al [9] using the generalized inequalities. However high-dimensional classification of multiple quantum systems still holds many puzzling open questions. In this paper using Bell function and the generalized three dimensional Pauli operators, we derive a quantum upper bound for

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3⊗3⊗3 quantum systems. Moreover we present a classification of entanglement for triqutrit mixed states by a set of Bell inequalities. These inequalities can distinguish fully separable and bi-separable states.

Consider two observers and allow each of them to choose independently between two dichotomic observables denoted here A_i and B_i for the *i-th* observer. An operator $\hat{V}_i$ denotes the operation for a variable $V_i \in \{A_i, B_i\}$ of *i-th* observer. We choose a complete set of orthonormal basis vectors $|k>$ to describe an operation of a given variable V_i. The operation outcomes are indicated by a set of eigenvalues $\{1, \omega, \omega^2\}$, where $\omega = \exp(\frac{2\pi i}{3})$ is a primitive third root of unity. Therefore the operator can be represented by $\hat{V}_i = \sum_{k=0}^{2} \omega^k |k><k|$. The Bell function [10] can be written as

$$\beta = \sum_{j=1}^{2} \frac{1}{4} (\hat{A_1}^j \otimes \hat{A_2}^j \otimes \hat{A_3}^j - \omega^j \hat{A_1}^j \otimes \hat{B_2}^j \otimes \hat{B_3}^j + \omega^j \hat{B_1}^j \otimes \hat{A_2}^j \otimes \hat{B_3}^j + \omega^j \hat{B_1}^j \otimes \hat{B_2}^j \otimes \hat{A_3}^j),$$

where $\hat{A_i}^j$ (or $\hat{B_i}^j$) denotes the *j-th* power of $\hat{A_i}$ (or $\hat{B_i}$).

Next we shall construct three Bell functions in terms of the above equation. Consider three dimensional Pauli operators [11] $\hat{X}$ and $\hat{Z}$ which satisfy

$$\hat{X}|k>=|k+1>, \quad \hat{Z}|k>=\omega^k|k>, \quad \hat{X}^3=I, \quad \hat{Z}^3=I,$$

where I denotes identity operator. Therefore we can replace $\hat{A_i}$ and $\hat{B_i}$ with the following unitary operators

$$\hat{A_1} = \hat{Z}, \ \hat{A_2} = \omega^2 \hat{X}\hat{Z}, \ \hat{A_3} = \hat{X}\hat{Z}^2,$$

$$\hat{B_1} = \hat{Z}, \ \hat{B_2} = \hat{X}\hat{Z}^2, \ \hat{B_3} = \omega^2 \hat{X}\hat{Z},$$

we can obtain

$$\beta_1 = \sum_{j=1}^{2} \frac{1}{4} (\hat{Z}^j \otimes (\omega^2 \hat{X}\hat{Z})^j \otimes (\hat{X}\hat{Z}^2)^j + \omega^j \hat{Z}^j \otimes (\hat{X}\hat{Z}^2)^j \otimes (\omega^2 \hat{X}\hat{Z})^j + \omega^j \hat{Z}^j \otimes (\omega^2 \hat{X}\hat{Z})^j \otimes (\omega^2 \hat{X}\hat{Z})^j + \omega^j \hat{Z}^j \otimes (\hat{X}\hat{Z}^2)^j \otimes (\hat{X}\hat{Z}^2)^j),$$

If we choose unitary operators as follows

$$\hat{A_1} = \omega^2 \hat{X}\hat{Z}, \ \hat{A_2} = \hat{X}\hat{Z}^2, \ \hat{A_3} = \hat{Z}, \ \hat{B_1} = \hat{X}\hat{Z}^2, \ \hat{B_2} = \omega^2 \hat{X}\hat{Z}, \ \hat{B_3} = \hat{Z},$$

we can obtain

$$\beta_2 = \sum_{j=1}^{2} \frac{1}{4} ((\omega^2 \hat{X}\hat{Z})^j \otimes (\hat{X}\hat{Z}^2)^j \otimes \hat{Z}^j + \omega^j (\omega^2 \hat{X}\hat{Z})^j \otimes (\omega^2 \hat{X}\hat{Z})^j \otimes \hat{Z}^j + \omega^j (\hat{X}\hat{Z}^2)^j \otimes (\hat{X}\hat{Z}^2)^j \otimes \hat{Z}^j + \omega^j (\hat{X}\hat{Z}^2)^j \otimes (\omega^2 \hat{X}\hat{Z})^j \otimes \hat{Z}^j),$$

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If we choose unitary operators as follows

$$\widehat{A_1} = \omega^2 \hat{X}\hat{Z},\ \widehat{A_2} = \hat{Z},\ \widehat{A_3} = \hat{X}\hat{Z}^2,\ \widehat{B_1} = \hat{X}\hat{Z}^2, \widehat{B_2} = \hat{Z},\ \widehat{B_3} = \omega^2 \hat{X}\hat{Z},$$

we can obtain

$$\beta_3 = \sum_{j=1}^{2} \frac{1}{4}((\omega^2 \hat{X}\hat{Z})^j \otimes \hat{Z}^j \otimes (\hat{X}\hat{Z}^2)^j + \omega^j (\omega^2 \hat{X}\hat{Z})^j \otimes \hat{Z}^j \otimes (\omega^2 \hat{X}\hat{Z})^j$$

$$+\omega^j (\hat{X}\hat{Z}^2)^j \otimes \hat{Z}^j \otimes (\omega^2 \hat{X}\hat{Z})^j + \omega^j (\hat{X}\hat{Z}^2)^j \otimes \hat{Z}^j \otimes (\hat{X}\hat{Z}^2)^j),$$

Now we consider the bound of $|\langle \beta_i \rangle|$ with i=1, 2, 3.

Theorem 1 For triqutrit mixed states, we have the inequality $|\langle \beta_i \rangle| \leq \frac{5}{4}$ with i=1,2, 3.

Proof . Due to the linear property of average values, it is sufficient to consider pure states only. The triqutrit pure state can be written as [12]

$$|\varphi\rangle = c_1|000\rangle + c_2|011\rangle +$$

$$c_3|012\rangle + c_4|021\rangle c_5|022\rangle + c_6|101\rangle$$

$$+c_7|102\rangle + c_8|110\rangle +$$

$$c_9|111\rangle + c_{10}|120\rangle c_{11}|122\rangle + c_{12}|201\rangle$$

$$+c_{13}|202\rangle + c_{14}|210\rangle +$$

$$c_{15}|212\rangle + c_{16}|220\rangle c_{17}|221\rangle + c_{18}\ \ 222\rangle,$$

where c_5, c_{11}, c_{13}, c_{15}, c_{16}, c_{17} and c_{18} are real and non-negative, $|c_1| \geq |c_i|$ with i=1, 2, ⋯, 18, $|c_9| \geq |c_{18}|$ and $\sum_{i=1}^{18} |c_i|^2 = 1$.

$$|\langle \beta_1 \rangle| = \left|\frac{5}{4}(-c_1c_2 + 5c_1c_5 - c_2c_5 - c_6c_{10} + 2c_7c_8 + 2c_9c_{11}\right.$$

.

$$\left.+2c_{12}c_{15} + 5c_{12}c_{16} - c_{13}c_{14} - c_{13}c_{17} - 4c_{14}c_{17} + 5c_{15}c_{16})\right|$$

$$\leq \frac{1}{8} \times 10 \times \sum_{i=1}^{18} |c_i|^2$$

$$= \frac{5}{4}$$

Similarly we can prove $|\langle \beta_i \rangle| \leq \frac{5}{4}$, with i=2,3. □

Next we consider the classification of entanglement for triqutrit mixed states.

Theorem 2 Suppose the triqutrit mixed state ρ is fully separable, then $|\langle \beta_i \rangle| = 0$ with i=1, 2, 3.

Proof. Due to the linear property of average values, it is sufficient to consider pure states only. The fully separable pure states via a suitable choice of bases can be written as

$$|\varphi\rangle = |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle$$

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therefore $|\langle\beta_i\rangle| = |\mathrm{tr}(\rho\beta_i)| = 0$. □

Theorem 3

For bi-separable states ρ_{1-23}, ρ_{12-3} and ρ_{13-2}, respectively, we have

$$|\langle\beta_1\rangle| \le \frac{3}{4}, \quad |\langle\beta_2\rangle| = 0, \quad |\langle\beta_3\rangle| = 0,$$

$$|\langle\beta_1\rangle| = 0 \quad |\langle\beta_2\rangle| \le \frac{3}{4}, \quad |\langle\beta_3\rangle| = 0,$$

$$|\langle\beta_1\rangle| = 0, \quad |\langle\beta_2\rangle| = 0, \quad |\langle\beta_3\rangle| \le \frac{3}{4}.$$

Proof. Due to the linear property of average values, it is sufficient to consider pure states only. Every pure state ρ_{1-23} via a suitable choice of bases can be written as [12]

$$|\varphi\rangle = |0\rangle\otimes(c_0|00\rangle + c_1|11\rangle + c_2|22\rangle),$$

where $|c_0| \ge |c_1| \ge |c_2|$ and $\sum_{i=0}^{2}|c_i|^2 = 1$.

Therefore

$$|\langle\beta_1\rangle| = \left|\frac{1}{4}(5c_2c_0 - c_0c_1 - c_1c_2)\right| \le \frac{3}{4}.$$

It is straightforward to prove the following results $\langle\beta_2\rangle| = 0$. $|\langle\beta_3\rangle| = 0$. For bi-separable states $\rho_{12-3}$and ρ_{13-2}, the results can be proved similarly. □

Remark If we consider $|\langle\beta_i\rangle|$ with i=1, 2, 3 to be three coordinates, all the triqutrit states are confined in a cube with size $\frac{5}{4}\times\frac{5}{4}\times\frac{5}{4}$. The bi-separable states are confined in a cube with size $\frac{3}{4}\times\frac{3}{4}\times\frac{3}{4}$.

We have presented a quantum upper bound for triqutrit mixed states using generalized Bell function and the generalized three dimensional pauli operators. A classification of triqutrit entanglement has been presented. These inequalities can distinguish fully separable states and three types of bi-separable states for the triqutrit states. Moreover the bi-separable states are in a cuboid with the size $\frac{3}{4}\times\frac{3}{4}\times\frac{3}{4}$. The triqutrits states are confined in a cube with size $\frac{5}{4}\times\frac{5}{4}\times\frac{5}{4}$.

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