Journal of Computational Science & Engineering 4 (2013) 161-169

Journal of Computational Science & Engineering

Available at http://www.asocse.org ISSN 1710-4068

A Fully Implicit Finite Difference Scheme for the Variable-Order Fractional Advection-Diffusion Equation with a Nonlinear Source Term

Weiyuan Ma*, Yabin Shao and Haidong Zhang

School of Mathematics and Computer Science, Northwest University for Nationalities, Lanzhou 73000, China

Article Information

Article history:
Received 3 January 2013
Revised 11 January 2013
Accepted 20 January 2013
Available online 30 January 2013

Keywords:
Variable-order nonlinear fractional diffusion equation
Fully implicit difference scheme
Iterative algorithm; Convergence
Stability

Abstract

This paper proposes a finite difference scheme for variable-order nonlinear fractional diffusion equation, in which the nonlinear source term is considered implicit. A linear iterative algorithm for nonlinear scheme is introduced. The stability and convergence of the method are proved by discrete energy method. The convergence order of the method is $O(\tau + h)$. Some examples demonstrate that the method is efficient and reliable.

1. Introduction

The use of constant-order fractional diffusion equations to characterize anomalous diffusion phenomena has received tremendous success. However, in some complicated situation, the constant-order fractional diffusion equations are not capable of characterizing diffusion processes, for instance, diffusion process in inhomogeneous or heterogeneous medium [1]. It has become clear that further theoretical and numerical investigations are required in order to incorporate adequate tool for the description of more complicated (or more realistic) stochastic diffusion processes. Variable-order differential operator is proposed to describe the anomalous diffusion process in which diffusion rate changes in the course of space or

* Corresponding author: mwy2004@126.com

2013 © The American Computational Science Society. All rights reserved.

time [2,3]. Variable-order differential operator model can be employed as a powerful tool in complex anomalous diffusion modeling [4-6].

Until now, a few studies of numerical methods have been down for the variable-order fractional differential equation. R. Lin et al. [7] constructed the explicit difference scheme for the variable-order nonlinear fractional diffusion equation. Zhuang et al. [8] proposed numerical methods for the variable-order fractional advection diffusion equation with a nonlinear source term. Chen et al. [9-12] focused on some high accuracy numerical methods for the different variable-order fractional differential equations with a nonlinear source term. Recently, Sun et al. [13] considered the numerical schemes for variable-order time fractional diffusion equation. It is should be noted that all of the above papers made the nonlinear source term linearized. So, fully implicit finite difference scheme for the fractional differential equation is indispensable. In this paper, a fully implicit finite difference scheme for the variable-order nonlinear fractional diffusion equation is proposed.

An outline of the remaining part of the article is as follows: In section 2, we construct a new finite difference method for the variable-order nonlinear fractional diffusion equation. In section 3, an iterative algorithm for the difference scheme is proposed. In section 4, we proved that difference scheme is stability and convergent without any restrictions on the mesh sizes. Finally, some numerical results are provided in section 5, they are in agreement with our theoretical analysis.

2. Derivation of the difference scheme

In this paper, we consider the following variable-order nonlinear fractional diffusion equation:

$$\frac{\partial u(x,t)}{\partial t}=k(x,t)R_{\alpha(x,t)}u(x,t)-v(x,t)\frac{\partial u(x,t)}{\partial t}+f(u,x,t),\quad (x,t)\in[a,b]\times[0,T], \tag{2.1}$$

With initial condition and boundary conditions

$$u(x,t)=u_0(x),\quad a\le x\le b, \tag{2.2}$$

$$u(a,t)=0,u(b,t)=0,\quad 0\le t\le T, \tag{2.3}$$

Where $1\le\underline{\alpha}\le\alpha(x,t)\le\bar{\alpha}\le 2;v(x,t)((0\le v(x,t)\le\bar{v})$ represents the average fluid velocity, $f(u,x,t)$ is a source /sink term that satisfies the Lipschitz condition $|f(u_1,x,t)-f(u_2,x,t)|\le L|u_1-u_2|$, and $0\le k(x,t)\le k.$

In (1.1), $R_{\alpha(x,t)}u(x,t)$ denotes variable-order fractional operator, which is defined as

$$R_{\alpha(x,t)}u(x,t)=\frac{1}{\Gamma(2-\alpha(x,t))}\times\left(c_+(x,t)\frac{d^2}{d\theta^2}\int_a^\theta\frac{u(\xi,t)d\xi}{(\theta-\xi)^{\alpha(x,t)-1}}+c_-(x,t)\frac{d^2}{d\theta^2}\int_\theta^b\frac{u(\xi,t)d\xi}{(\xi-\theta)^{\alpha(x,t)-1}}\right)_{\theta=x} \tag{2.4}$$

where $0\le c_+(x,t)\le c_1,0\le c_-(x,t)\le c_2.$

For the special case $c_+(x,t)=1,c_-(x,t)\equiv 0.$ $R_{\alpha(x,t)}u(x,t)$ is found to be Riemann-Liouville left-handed spatial fractional derivative; the particular case for $c_+(x,t)\equiv 0,\quad c_-(x,t)=0$, $R_{\alpha(x,t)}u(x,t)$ becomes Riemann-Liouville right-handed spatial fractional derivative;

$$c_+(x,t)=c_-(x,t)=-\frac{1}{2\cos(\frac{\pi\alpha(x,t)}{2})}$$

when , $R_{\alpha(x,t)}u(x,t)$ represents the fractional derivative. The classical advection-diffusion equation with a

2013 © The American Computational Science Society. All rights reserved.

nonlinear source term:

$$\frac{\partial u(x,t)}{\partial t}=k(x,t)\frac{\partial^2 u(x,t)}{\partial x^2}-v(x,t)\frac{\partial u(x,t)}{\partial x}+f(u,v,t)$$

Corresponds to $\alpha(x,t)=2$.

We take an equally spaced mesh of J points for the spatial domain $a\le x\le b$, N constant time steps for the temporal domain $0\le t\le T$, and denote the spatial grid points by

$$x_j=jh, j=0,1,\cdots,J,$$

and the temporal grid point by $t_n=n\tau, n=0,1,\cdots,N,$

where the grid spacing is simply $h=(b-a)/J$ in the spatial domain and $\tau=T/N$ in the temporal domain. We introduce the following notations:

$$\Omega=\{(x,t)\mid a\le x\le b, 0\le t\le T\},$$

$$G(\Omega)=\left\{u(x,t)\mid \frac{\partial^4 u(x,t)}{\partial x^4},\frac{\partial^2 u(x,t)}{\partial t^2}\in C(\Omega)\right\}.$$

In this paper, we suppose the continuous problem (2.1)-(2.3) has a smooth solution $u(x,t)\in G(\Omega)$.

Adopting the discrete scheme in [7, 8], we discrete the formula in variable-order fractional operator as

$$\frac{1}{\Gamma(2-\alpha(x_j,t_{n+1}))}\left(\frac{d^2}{d\theta^2}\int_a^\theta\frac{u(\xi,t)d\xi}{(\theta-\xi)^{\alpha(x_j,t_{n+1})-1}}\right)_{\theta=x_j}$$
$$=h^{-\alpha_{j+1}^{n+1}}\sum_{k=0}^{j+1}\omega_{j+1-k}^{(\alpha_{j+1}^{n+1})}u(x_j,t_{n+1})+O(h),$$

(2.5)

$$\frac{1}{\Gamma(2-\alpha(x_j,t_{n+1}))}\left(\frac{d^2}{d\theta^2}\int_\theta^b\frac{u(\xi,t)d\xi}{(\xi-\theta)^{\alpha(x_j,t_{n+1})-1}}\right)_{\theta=x_j}$$
$$=h^{-\alpha_{j-1}^{n+1}}\sum_{k=j-1}^{J}\omega_{k-j+1}^{(\alpha_{j-1}^{n+1})}u(x_j,t_{n+1})+O(h),$$

(2.6)

where $\omega_k^{(\alpha_j^{n+1})}=(-1)^k\binom{\alpha_j^{n+1}}{k}$, can be evaluated recursively

$$\omega_0^{\alpha_j^{n+1}}=1,\omega_k^{\alpha_j^{n+1}}=(1-\frac{\alpha_j^{n+1}+1}{k})\omega_{k-1}^{\alpha_j^{n+1}},$$

and $\alpha_j^{n+1}=\alpha(x_j,t^{n+1}), k=2,3,\cdots.$

We propose the following fully implicit finite difference scheme for the solution of (2.1)-(2.3):

$$\frac{u_j^{n+1}-u_j^n}{\tau}$$
$$=k_j^{n+1}\left(c_{+j}^{n+1}h^{-\alpha_{j+1}^{n+1}}\sum_{k=0}^{j+1}\omega_{j+1-k}^{(\alpha_{j+1}^{n+1})}u_k^{n+1}+c_{-j}^{n+1}h^{-\alpha_{j-1}^{n+1}}\sum_{k=j-1}^{J}\omega_{k-j+1}^{(\alpha_{j-1}^{n+1})}u_k^{n+1}\right)$$
$$-v_j^{n+1}\frac{u_j^{n+1}-u_{j-1}^{n+1}}{h}+f_j^{n+1},$$
$$j=1,2,\cdots,J-1, n=0,1,\cdots,N-1$$

(2.7)

$$u_j^0=u_0(x_j), \quad j=1,2,\cdots,J-1,$$

(2.8)

$$u_0^n=0, u_J^n=0, \quad n=0,1,\cdots,N.$$

(2.9)

where

$$k_j^{n+1}=k(x_j,t^{n+1}), c_{+j}^{n+1}=c_+(x_j,t^{n+1}),$$
$$c_{-j}^{n+1}=c_-(x_j,t^{n+1})$$

$f_j^{n+1}=f(u_j^{n+1},x_j,t^{n+1})$, and u_j^n the approximation of $u(x_j,t_n)$.

The above equation (2.7) can be simplified as follows:

2013 © The American Computational Science Society. All rights reserved.

$$
\begin{aligned}
&-(\xi_j^{n+1}\omega_2^{(\alpha_{j+1}^{n+1})}+\eta_j^{n+1}\omega_0^{(\alpha_{j-1}^{n+1})}+\zeta_j^{n+1})u_{j-1}^{n+1}\\
&-\eta_j^{n+1}\sum_{k=j+2}^{J}\omega_{k-j+1}^{(\alpha_{j-1}^{n+1})}u_k^{n+1}\\
&-(\xi_j^{n+1}\omega_0^{(\alpha_{j+1}^{n+1})}+\eta_j^{n+1}\omega_2^{(\alpha_{j-1}^{n+1})})u_{j+1}^{n+1}\\
&-\xi_j^{n+1}\sum_{k=0}^{j-2}\omega_{j+1-k}^{(\alpha_{j+1}^{n+1})}u_k^{n+1}\\
&+[1+\zeta_j^{n+1}-(\xi_j^{n+1}\omega_1^{(\alpha_{j+1}^{n+1})}+\eta_j^{n+1}\omega_1^{(\alpha_{j-1}^{n+1})})]u_j^{n+1}\\
&=u_j^n+\tau f_j^{n+1},
\end{aligned}
\tag{2.10}
$$

for $j=1,2,\cdots,J-1, n=0,1,\cdots,N-1,$ where $\xi_j^{n+1}=\tau k_j^{n+1}c_{+j}^{n+1}h^{-\alpha_{j+1}^{n+1}}, \eta_j^{n+1}=\tau k_j^{n+1}c_{-j}^{n+1}h^{-\alpha_{j-1}^{n+1}}$, $\zeta_j^{n+1}=\tau v_j^{n+1}/h$.

3. Iterative algorithm

It is obvious that to solve the system of finite difference scheme (2.7) requires solving a nonlinear system at each time level with $J-1$ unknowns. The cost of solving a nonlinear system is too high. So, we propose the following linear iterative method to solve it.

For $n=0,1,\cdots,N-1,$

$$
\begin{aligned}
&u_j^{n+1(m)}-u_j^n=\xi_j^{n+1}\sum_{k=0}^{j+1}\omega_{j+1-k}^{(\alpha_{j+1}^{n+1})}u_k^{n+1(m)}\\
&+\eta_j^{n+1}\sum_{k=j-1}^{J}\omega_{k-j+1}^{(\alpha_{j-1}^{n+1})}u_k^{n+1(m)}\\
&-\zeta_j^{n+1}(u_j^{n+1(m)}-u_{j-i}^{n+1(m)})\\
&+\tau f(u_j^{n+1(m-1)},x_j,t^{n+1}),
\end{aligned}
\tag{3.1}
$$

with

$$
u_j^{n+1(0)}=\begin{cases}u_j^n, & n=0,\\ 2u_j^n-u_j^{n-1}, & n\le 1.\end{cases}
\tag{3.2}
$$

If for a given tolerance ε, $\| u^{n+1(m)}-u^{n+1(m-1)} \|_\infty \le \varepsilon$, then we accept $u_j^{n+1(m)}$ as u_j^{n+1} at the time level $n+1$.

4. Stability and convergence of the numerical method

In this section, we will analyze the stability of the numerical scheme (2.7). Let $u_i^n, \tilde{u}_i^n \ (i=1,2,\cdots,J-1, n=0,1,2,\cdots,N-1)$ be the solution of difference scheme (2.10) with initial value u_i^0 and $\tilde{u}_i^0$, respectively. Thereby, the error $\varepsilon_i^n=\tilde{u}_i^n-u_i^n$ satisfy

$$
\begin{aligned}
&-(\xi_j^{n+1}\omega_2^{(\alpha_{j+1}^{n+1})}+\eta_j^{n+1}\omega_0^{(\alpha_{j-1}^{n+1})}+\zeta_j^{n+1})\varepsilon_{j-1}^{n+1}\\
&-\xi_j^{n+1}\sum_{k=0}^{j-2}\omega_{j+1-k}^{(\alpha_{j+1}^{n+1})}\varepsilon_k^{n+1}\\
&-(\xi_j^{n+1}\omega_0^{(\alpha_{j+1}^{n+1})}+\eta_j^{n+1}\omega_2^{(\alpha_{j-1}^{n+1})})\varepsilon_{j+1}^{n+1}\\
&+[1+\zeta_j^{n+1}-(\xi_j^{n+1}\omega_1^{(\alpha_{j+1}^{n+1})}+\eta_j^{n+1}\omega_1^{(\alpha_{j-1}^{n+1})})]\xi_j^{n+1}\\
&-\eta_j^{n+1}\sum_{k=j+2}^{J}\omega_{k-j+1}^{(\alpha_{j-1}^{n+1})}\varepsilon_k^{n+1}\\
&=\varepsilon_j^n+\tau[f(\tilde{u}_j^{n+1},x_j,t^{n+1})-f(u_j^{n+1},x_j,t^{n+1})].
\end{aligned}
\tag{4.1}
$$

To show the stability, we shall use the following lemma.

Lemma 4.1 ([7,8]) The variable-order Grünwald coefficients

$$\omega_k^{(\alpha(x,t))}=(-1)^k\binom{\alpha(x,t)}{k},$$

$1<\alpha(x,t)<2, k=0,1,\cdots$, satisfy the conditions $(1)\omega_0^{(\alpha(x,t))}=1, \omega_1^{(\alpha(x,t))}=\alpha(x,t).$

2013 © The American Computational Science Society. All rights reserved.

$$(2)\omega_0^{\alpha(x,t)}$$
$$=(-1)^k\frac{\alpha(x,t)(\alpha(x,t)-1)\cdots(\alpha(x,t)-k+1)}{k!}>0,$$
$$k=2,3,\cdots.$$
$$(3)\sum_{k=0}^{\infty}\omega_k^{\alpha(x,t)}=0.$$
$$(4)\ \sum_{k=0,k\neq 1}^{m}\omega_k^{\alpha(x,t)}\le-\omega_1^{\alpha(x,t)},\qquad m=1,2,\cdots.$$

We now consider the stability of numerical scheme (2.7).

Theorem 4.1 For small τ, the solution of the fractional implicit difference scheme (2.7)-(2.9) is stable for initial data in norm $\|\cdot\|_\infty$.

Proof. The proof of difference scheme is stable equivalent to proof the following inequality:

$$\|E^n\|_\infty\le C\|E^0\|_\infty, n=0,1,\cdots,N-1,$$

where $E^n=(\varepsilon_1^k,\varepsilon_2^k,\cdots,\varepsilon_{J-1}^k).$

Suppose $|\varepsilon_l^{n+1}|=\max_{1\le i\le J-1}|\varepsilon_i^{n+1}|$, using lemma 4.1 and (4.1), we have

$$\|E^{n+1}\|_\infty=|\varepsilon_l^{n+1}|$$
$$\le\{-(\xi_l^{n+1}\omega_2^{(\alpha_{j+1}^{n+1})}+\eta_l^{n+1}\omega_0^{(\alpha_{l-1}^{n+1})})$$
$$+[1-(\xi_l^{n+1}\omega_1^{(\alpha_{l+1}^{n+1})}+\eta_l^{n+1}\omega_l^{(\alpha_{j-1}^{n+1})})]$$
$$\frac{1}{\Gamma(2-\alpha(x_j,t_{n+1}))}\left(\frac{d^2}{d\theta^2}\int_a^\theta\frac{u(\xi,t)d\xi}{(\theta-\xi)^{\alpha(x_j,t_{n+1})-1}}\right)_{\theta=x_j}$$
$$=h^{-\alpha_{j+1}^{n+1}}\sum_{k=0}^{j+1}\omega_{j+1-k}^{(\alpha_{j+1}^{n+1})}u(x_j,t_{n+1})+O(h),$$
$$-(\xi_l^{n+1}\omega_0^{(\alpha_{l+1}^{n+1})}+\eta_l^{n+1}\omega_2^{(\alpha_{l-1}^{n+1})})$$
$$-\xi_l^{n+1}\sum_{k=0}^{l-2}\omega_{l+1-k}^{(\alpha_{l+1}^{n+1})}-\eta_l^{n+1}\sum_{k=l+2}^{J}\omega_{k-l+1}^{(\alpha_{l-1}^{n+1})}\}|\varepsilon_l^{n+1}|$$
$$\le-(\xi_l^{n+1}\omega_2^{(\alpha_{l+1}^{n+1})}+\eta_l^{n+1}\omega_0^{(\alpha_{l-1}^{n+1})}+\zeta_l^{n+1})|\varepsilon_{l-1}^{n+1}|$$
$$-\xi_l^{n+1}\sum_{k=0}^{l-2}\omega_{l+1-k}^{(\alpha_{l+1}^{n+1})}|\varepsilon_k^{n+1}|$$
$$-(\xi_l^{n+1}\omega_0^{(\alpha_{l+1}^{n+1})}+\eta_l^{n+1}\omega_2^{(\alpha_{l-1}^{n+1})})|\varepsilon_{l+1}^{n+1}|$$
$$-\eta_l^{n+1}\sum_{k=l+2}^{J}\omega_{k-l+1}^{(\alpha_{l-1}^{n+1})}|\varepsilon_k^{n+1}|$$
$$+[1+\zeta_l^{n+1}-(\xi_l^{n+1}\omega_1^{(\alpha_{l+1}^{n+1})}+\eta_l^{n+1}\omega_1^{(\alpha_{l-1}^{n+1})})]|\varepsilon_l^{n+1}|$$
$$\le|-(\xi_l^{n+1}\omega_2^{(\alpha_{l+1}^{n+1})}+\eta_l^{n+1}\omega_0^{(\alpha_{l-1}^{n+1})}+\zeta_l^{n+1})\varepsilon_{l-1}^{n+1}$$
$$-\xi_l^{n+1}\sum_{k=0}^{l-2}\omega_{l+1-k}^{(\alpha_{l+1}^{n+1})}\varepsilon_k^{n+1}$$
$$-(\xi_l^{n+1}\omega_0^{(\alpha_{l+1}^{n+1})}+\eta_l^{n+1}\omega_2^{(\alpha_{l-1}^{n+1})})\varepsilon_{l+1}^{n+1}$$
$$-\eta_l^{n+1}\sum_{k=l+2}^{J}\omega_{k-l+1}^{(\alpha_{l-1}^{n+1})}\varepsilon_k^{n+1}$$
$$+[1+\zeta_l^{n+1}-(\xi_l^{n+1}\omega_1^{(\alpha_{l+1}^{n+1})}+\eta_l^{n+1}\omega_1^{(\alpha_{l-1}^{n+1})})]\varepsilon_l^{n+1}|$$
$$\le\ |\varepsilon_l^n|+\tau|f(\tilde{u}_l^{n+1},x_l,t^{n+1})-f(u_l^{n+1},x_l,t^{n+1})|$$
$$\le\ \|E^n\|_\infty+\tau L\|E^{n+1}\|_\infty. \qquad (4.2)$$

When τ is small enough to satisfy $\tau\le\frac{1}{2L}$, (4.2) can be expressed as

$$\|E^{n+1}\|_\infty\le\frac{1}{1-\tau L}\|E^n\|_\infty\le(1+2\tau L)\|E^n\|_\infty$$
$$\le\cdots\le(1+2\tau L)^n\|E^n\|_\infty\le e^{2LT}\|E^n\|_\infty$$
$$=C\|E^n\|_\infty. \qquad (4.3)$$

Hence we obtain the desired result.

We now show that the approximate solutions u_j^n of (2.7)-(2.9) converge to the exact solution $u(x_j,t^n)$ of (1.1).

Theorem 4.2 Under the conditions of Theorem 1, for small τ, the difference scheme (2.7)-(2.9) is convergence in discrete L_∞-norm with the convergence rate of order $O(\tau+h)$.

2013 © The American Computational Science Society. All rights reserved.

Proof. Suppose $e_i^k = u(x_i, t_k) - u_i^k$ and $e^k = (e_1^k, e_2^k, \cdots, e_{J-1}^k)^T$. From (2.1), (2.2) and Taylor expansion, we have:

$$
\begin{aligned}
&-(\xi_j^{n+1}\omega_2^{(\alpha_{j+1}^{n+1})} + \eta_j^{n+1}\omega_0^{(\alpha_{j-1}^{n+1})} + \zeta_l^{n+1})e_{j-1}^{n+1} \\
&-\eta_j^{n+1}\sum_{k=j+2}^{J}\omega_{k-j+1}^{(\alpha_{j-1}^{n+1})}e_k^{n+1} \\
&-(\xi_j^{n+1}\omega_0^{(\alpha_{j+1}^{n+1})} + \eta_j^{n+1}\omega_2^{(\alpha_{j-1}^{n+1})})e_{j+1}^{n+1} \\
&-\xi_j^{n+1}\sum_{k=0}^{j-2}\omega_{j+1-k}^{(\alpha_{j+1}^{n+1})}e_k^{n+1} \\
&+[1+\zeta_l^{n+1} - (\xi_j^{n+1}\omega_1^{(\alpha_{j+1}^{n+1})} + \eta_j^{n+1}\omega_1^{(\alpha_{j-1}^{n+1})})]e_j^{n+1} \\
&= e_j^n + \tau[f(u(x_j,t^{n+1}),x_j,t^{n+1}) - f(u_j^{n+1},x_j,t^{n+1})] \\
&+\tau R_j^n,
\end{aligned}
\tag{4.4}
$$

where R_j^n is the truncation errors of the difference scheme (2.7) and there exists a positive constant C_1 such that $|R_j^n| \le C_1(\tau + h)$, C_1 independent of step sizes τ and h.

Set $|e_l^{n+1}| = \max_{1\le j\le J-1}|e_j^{n+1}|$, similar to (4.2), we obtain

$$
\|e^{n+1}\|_\infty = |e_l^{n+1}| \le \|e^n\|_\infty + \tau L\|e^{n+1}\|_\infty + \tau|R_j^n|. \tag{4.5}
$$

Applying $e_j^0 = 0$, and (4.5), we have

$$
\begin{aligned}
\|e^{n+1}\| &\le (1+2\tau L)\|e^n\|_\infty + C_1\tau(\tau+h) \\
&\le (1+2\tau L)^2\|e^{n-1}\|_\infty \\
&\quad + C_1\tau(\tau+h)((1+2\tau L)+1) \\
&\le \cdots \\
&\le (1+2\tau L)^{n+1}\|e^0\|_\infty \\
&\quad + C_1\tau(\tau+h)((1+2\tau L)^n + \cdots + 1) \\
&\le C_1\tau(\tau+h)(ne^{2LT}) = O(\tau+h).
\end{aligned}
\tag{4.6}
$$

This proves that u_j^n converges to $u(x_j, t^n)$ as h and τ tends to zero. Hence, the conclusion is obtained.

5. Numerical simulations

In order to demonstrate the effectiveness of our theoretical analysis, two numerical examples are presented.

Example 1 Consider the following variable-order nonlinear fractional advection-diffusion equation [8],

$$
\frac{\partial u}{\partial t} = -v(x,t)\frac{\partial u}{\partial x} + R_{\alpha(x,t)}u + f(u,x,t), \quad 0 < x < 1, 0 < t \le T, \tag{5.1}
$$

$$
u(0,t) = 0, u(1,t) = 0, \quad 0 < t \le T, \tag{5.2}
$$

$$
u(x,0) = x^2(1-x)^2, \quad 0 < x < 1, \tag{5.3}
$$

where

$$
\begin{aligned}
v(x,t) &= 6x^3(1-x)^3e^t \\
\alpha(x,t) &= 1.5 + 0.4\sin(0.5\pi x t), \\
c_+(x,t) &= 0.5\Gamma(5-\alpha(x,t))x^{2+\alpha(x,t)}(1-x)^4e^t, \\
c_-(x,t) &= 0.5\Gamma(5-\alpha(x,t))x^4(1-x)^{2+\alpha(x,t)}e^t, \\
f(u,x,t) &= u + u^2(-24x^2 + 2\alpha(x,t)(4-\alpha(x,t))).
\end{aligned}
$$

The solution of the above problem is $u(x,t) = e^t x^2(1-x)^2$.

The computational results are shown in the following figure and table. Fig. 1 shows exact solutions and numerical solutions obtained by the fully implicit finite difference method (2.7)-(2.9) with $h = 1/2, \tau = 1/100, \delta = 10^{-8}$ at three different times $T = 1, T = 1.5, T = 2$. The numerical solutions compare well with the exact solutions, which proved that the fully implicit finite difference method is stable. Table 1 shows the estimate of error $\|e^N\|_\infty = \max_{1\le j\le J-1}|u_j^n - u(x_j,t^n)|$ for various step sizes h and τ. According to Table 1, we can see that the computational results are getting better as h and τ become smaller. Therefore, Table 1 confirms the convergence of

2013 © The American Computational Science Society. All rights reserved.

order $O(\tau+h)$ in L_∞-norm of the fully implicit difference scheme for variable-order nonlinear fractional diffusion equation.

Using the extrapolation technique [8,16] in implementation of the scheme (2.7)-(2.9) for solving Example 1, one can obtain high accuracy.

The extrapolated solution is then computed from $u(x,t) \approx 2u_{2j}^{2n}(h/2,\tau/2) - u_j^n, j = 1,2,\cdots,(J+1)/2,$ where $t = t_n,\ x = x_j$ on the coarse grid, while $t = t_{2n}, x = x_{2j}$ on the fine grid, respectively. Table 2 shows the high accuracy of the scheme (2.7)-(2.9) when using the extrapolation technique, and the order of accuracy is $O(\tau^2 + h^2)$. The fully implicit finite difference with extrapolation technique is more accurate than semi implicit finite difference scheme [8] with extrapolation technique.

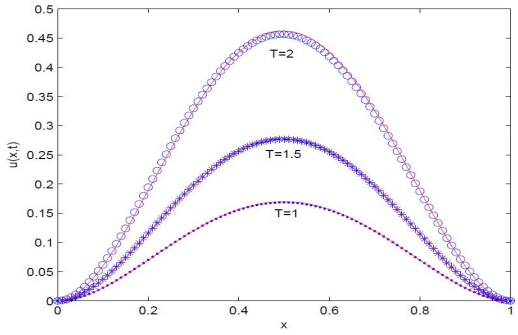

Fig. 1 The comparison of exact solutions and numerical solutions for solving Example 1 with h = 1/100 and τ =1/100 at three difference times

Table 1 $\left\|e^N\right\|_\infty$ computed by the difference scheme for solving Example 1 with T=1

$h=\tau$	$\left\|e^N\right\|_\infty$	$\left\|e^N(h,\tau)\right\|_\infty / \left\|e^N(\frac{h}{2},\frac{\tau}{2})\right\|_\infty$	$\left\|e^N(h,\tau)\right\|_\infty/(h^2+\tau^2)$
1/20	0.005	--	0.0567
1/40	0.0030	1.9000	0.0595
1/80	0.0015	2.0000	0.0609
1/160	7.7054e-004	1.9467	0.0616

Table 2 Comparison of the semi implicit (SI) and fully implicit (FI) finite difference scheme with extrapolation technique for solving Example 1 with T=1

$h=\tau$	Max error-SI	Max error-FI	Error-FI rate	Maxerror-FI/($h=\tau$)
1/20	7.844E-3	3.360E-4	--	0.2764
1/30	3.242E-3	1.526E-4	2.2670 ≈(30/20)2	0.2748
1/40	1.784E-3	8.650E-5	1.7646 ≈(40/30)2	0.2768
1/50	1.134E-3	5.532E-5	1.5638 ≈(50/40)2	0.2766
1/60	7.853E-4	3.853E-5	1.4358 ≈(50/40)2	0.2774

Example 2 In order to numerically check thepretical properties in the previous section, we consider the variable-order nonlinear Riesz fractional advection-diffusion equation[8],

$$\frac{\partial u}{\partial t} = -v(x,t)\frac{\partial u}{\partial x} + k(x,t)R^{\alpha(x,t)}u(x,t) + f(u,x,t), \quad 0 < x < 1, 0 < t \le T, \tag{5.4}$$

$$u(0,t) = u(1,t) = 0, \quad 0 \le t \le T, \tag{5.5}$$

$$u(x,0) = x^2(1-x)^2, \quad 0 < x < 1, \tag{5.6}$$

where

$$v(x,t) = x^3(1-x)^3 e^t,$$

$$\alpha(x,t) = 1.5 + 0.4\sin(0.5\pi x t),$$

$$c_+(x,t) = c_-(x,t) = -\frac{1}{2\cos(0.5\pi\alpha(x,t))},$$

2013 © The American Computational Science Society. All rights reserved.

$$k(x,t) = -2\cos(0.5\pi\alpha(x,t))\Gamma(5-\alpha(x,t))(x-x^2)^{2+\alpha(x,t)},$$

$$g_1(x,t) = 2x^2(1-x)^{\alpha(x,t)}[12x^2 + (4-\alpha(x,t))(-6x+3-\alpha(x,t))],$$

$$g_2(x,t) = 2(1-x)^2 x^{\alpha(x,t)}[12(1-x)^2 + (4-\alpha(x,t))(6x-3-\alpha(x,t))]$$

$$f(u,x,t) = [1-g_1(x,t)-g_2(x,t)]u + 2(1-2x)u^2.$$

The above problem has the exact solution $u(x,t) = e^t x^2(1-x)^2$.

Table 3 shows the L_∞-errors and convergence ratios calculated by fully implicit finite difference method (2.7)-(2.9) with a steps $\tau = h = 1/100$ at time $T = 1$. It is observed that the numerical result coincides with the theoretical analysis. Table 4 shows the approximate solution of Example 2 that computed by the semi implicit finite difference scheme [8] and the fully implicit finite difference with $\tau = h = 1/100$ at four different times, respectively. It is seen that the fully implicit is more accurate than the semi implicit finite difference scheme.

Table 3 $\left\|e^N\right\|_\infty$ computed by the difference scheme for solving Example 2 with T=1

$h=\tau$	$\left\|e^N\right\|_\infty$	$\left\|e^N(h,\tau)\right\|_\infty / \left\|e^N(\frac{h}{2},\frac{\tau}{2})\right\|_\infty$	$\left\|e^N(h,\tau)\right\|_\infty /(h+\tau)$
1/20	0.0021	--	0.0210
1/40	0.0010	2.1000	0.0204
1/80	5.037E-4	1.9853	0.0201
1/160	2.503E-4	2.0125	0.0200

According to the data of the tables and figures, we can draw a conclusion that the fully implicit finite difference scheme is efficient and reliable to solve variable-order fractional diffusion equation.

Table 4 $\left\|e^N\right\|_\infty$ computed by the two difference schemes for solving Example 2 at difference T

Scheme	T=1	T=2	T=3	T=4
Semi implicit	0.0018	0.0109	0.0652	0.4669
fully implicit	4.020E-4	0.002	0.0123	0.2110

6. Conclusions

In this paper, fully implicit finite difference scheme for the variable-order fractional advection-diffusion equation with a nonlinear source is described and demonstrated. An iterative algorithm is proposed to solve nonlinear scheme. The stability and convergence of the fully implicit numerical methods are analyzed systematically. Finally, some examples are given to demonstrate the effectiveness of the theoretical analysis. This method and technique can be applied to solve variable-order fractional (in space and in time) partial differential equation with a nonlinear source term.

Acknowledgement

The authors are grateful for the financial supported by the National Natural Scientific Fund of China (11161041) and the Fundamental Research Funds for the Central Universities (zyz2012076).

Reference

[1] A.V. Chechkin, R. Gorenflo, I.M. Sokolov, J. Phys. A: Math. Gen., 38 (2009) L679-L684.

[2] C.F. Lorenzo, T.T. Hartley, NASA/TP, 208 (1998) 208415.

2013 © The American Computational Science Society. All rights reserved.

[3] C.F. Lorenzo, T.T. Hartley, Nonlinear Dyn., 29 (2002) 57-98.

[4] H. Sun, W. Chen, Y. Chen, Phys., A388 (2009) 4586-4592.

[5] M. Meerschaert, C. Tadjeran, J. Comput. Appl. Math., 172 (2004) 65-77.

[6] K.S. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, John Wiley, New York, 1993.

[7] R. Lin, F. Liu, V. Anh, I. Turner, Appl. Math. Comput., 212 (2009) 435-445.

[8] P. zhuang, F. Liu, V. Anh, I. Turner, SIAM J. Numer. Anal., 47 (2009) 1760-1781.

[9] C. Chen, F. Liu, V. Anh, I. Turner, SIAM J. Sci. Comput., 32(2010) 1740-1760.

[10] C. Chen, F. Liu, V. Anh, I. Turner, Appl. Math. Comput., 217 (2011) 5729-5742.

[11] C. Chen, F. Liu, K. Burrage, J. Comput. Appl. Math., 236 (2011) 209-224.

[12] C. Chen, F. Liu, I. Turner, V. Anh, Comput. Math. Appl., 62 (2011) 971-986.

[13] H. Sun, W. Chen, C. Li, Y. Chen, Int. J. Bifucat. Chaos, 22 (2012) 1250085(16 pages).

[14] C. Tadjeran, M.M. Meerschaert, H.P. Scheffer, J. Comput. Phys., 213 (2006) 205-213.

2013 © The American Computational Science Society. All rights reserved.